1-ma‘ruza. Mavzu: Ikki o’lchovli integral

Reja:

1. Ikki o’lchovli integral tushunchasiga olib keluvchi hajm haqidagi masala.
2. Ikki o’lchovli integralning ta‘rifi va mavjudligi.
3. Ikki o’lchovli integralning geometrik ma‘nosi.
4. Ikki o’lchovli integralning mexanik ma‘nosi.
5. Ikki o’lchovli integralning xossalari.
6. Ikki o’lchovli integralni hisoblash.

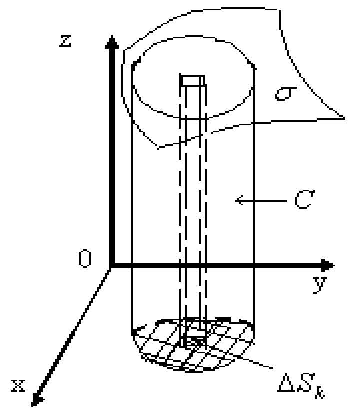
1.1. Ikki o’lchovli integral tushunchasiga olib keluvchi hajm haqidagi masala

$0xy$ tekislikda yopiq ℓ chiziq bilan chegaralangan D sohani hamda pastdan shu soha bilan, yon yoqdan yo‘naltiruvchisi D sohaning chegarasi ℓ egri chiziqdan iborat yasovchisi $0z$ o‘qqa parallel silindrik C sirt bilan, yuqoridan tenglamasi $z = f(x, y)$ bo‘lgan σ sirtning qismi (bo‘lagi) bilan chegaralangan silindrik jismni qaraymiz (1-chizma). Bunda $f(x, y)$ funksiya D sohada aniqlangan, uzluksiz va musbat deb faraz qilamiz. Shu silindrik jismning hajmini topish talab etilsin.

D sohani istalgan chiziqlar yordamida n ta ixtiyoriy kichik $\Delta S_1, \Delta S_2, ..., \Delta S_n$ yuzchalarga ajratamiz, bunda $\sum_{k=1}^{n} \Delta S_k = S - D$ sohaning yuzi.

Har bir ΔS_k yuzchani ustida yuqoridan σ sirtning bo‘lakchasi bilan chegaralangan va $0xy$ tekislikdagi proyeksiyasi ΔS_k dan iborat silindr yasaymiz. Bu bilan qaralayotgan asosiy silindrik jism n ta asosiy ΔS_k bo‘lgan ustunchalarga ajratiladi. Asosi ΔS_k bo‘lgan ustunchaning hajmini ΔV_k orqali belgilaymiz. U holda silindrik jismning hajmi V shu ustunchalarning hajmlarini yig‘indisiga teng:

$$V = \sum_{k=1}^{n} \Delta V_k .$$



1-chizma.

Asosi ΔS_k bo‘lgan silindrni qaraymiz. ΔS_k da ixtiyoriy $P_k(x_k, y_k)$ nuqtani olib bu silindrning balandligi sifatida σ sirtdagi P_k nuqtaga mos nuqtaning applikatasi $z_k = f(x_k, y_k)$ ni olamiz.

Bu silindrning hajmi asosining yuzi ΔS_k bilan balandligi $f(x_k, y_k)$ ni ko‘paytmasini asosi ΔS_k bo‘lgan ustunning hajmi ΔV_k ning taqribiy qiymati deb olamiz: $\Delta V_k \approx f(x_k, y_k)$.

Barcha shunaqa hajmlarning yig‘indisini olsak silindrik jismning hajmi V ni taqribiy qiymati kelib chiqadi:

$$V = \sum_{k=1}^{n} \Delta V_k \approx \sum_{k=1}^{n} f(x_k,\ y_l) \Delta S_k .$$

V ning aniq qiymati sifatida

$$\sum_{k=1}^{n} f(x_k, y_l)\Delta S_k$$

yig‘indining ΔS_k yuzchalarning soni cheksiz ortgandagi va har bir yuzchalar nuqtaga tortilgandagi (yig‘ilgandagi) limitini qabul qilamiz:

$$V = \lim_{\max diam\Delta S_k \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta S_k \qquad (1.1)$$

Shunday qilib, silindrik jismning hajmi V ni topish masalasi qandaydir limitni hisoblashga keltirildi.

1.2. Ikki o’lchovli integralning ta‘rifi va mavjudligi

Yuqoridagi $y = f(x) \geq 0$ uzluksiz egri chiziq, chapdan $x = a$, o’ngdan $x = b$ vertikal to’g’ri chiziqlar, quyidan $0x$ o’qning $[a, b]$ kesmasi bilan chegaralangan egri chiziqli trapetsiyaning yuzini topish masalasi $\sum_{k=1}^{n} f(x_k)\Delta x_k$ ko’rinishdagi yig’indining $\max \Delta x_k \to 0$ dagi limitini topishga, ya‘ni **aniq integral** tushunchasiga olib kelgan edi.

Shuningdek yuqoridan $z = f(x, y)$ sirt, quyidan $0xy$ tekislik va yasovchisi $0z$ o’qqa parallel, yo’naltiruvchisi D sohaning chegarasi yopiq ℓ egri chiziqdan iborat bo’lgan silindrik sirt bilan chegaralangan jismning hajmini topish masalasi $\sum_{k=1}^{n} f(x_k, y_l)\Delta S_k$ (1.1) ko’rinishdagi yig’indining ΔS_k sohalarning diametrlarini eng kattasi 0 ga intilgandagi limitini topishga, ya‘ni **ikki o’lchovli integral** tushunchasiga olib keldi.

$0xy$ tekislikda yopiq ℓ chiziq bilan chegaralangan D sohani hamda bu sohada aniqlangan va chegaralangan $z = f(x, y)$ funksiyani qaraymiz.

D sohani ixtiyoriy chiziqlar bilan n ta umumiy ichki nuqtalarga ega bo‘lmagan $\Delta S_1, \Delta S_2, ..., \Delta S_n$ bo’laklarga bo’lib ularni elementar yuzchalar deb ataymiz va ularning yuzlarini ham $\Delta S_1, \Delta S_2, ..., \Delta S_n$ orqali belgilaymiz. ΔS_k $(k = \overline{1, n})$ yuzlarning har birida bittadan nuqta olib ularni $P_1, P_2, ..., P_n$ lar orqali belgilaymiz. Funksiyaning tanlangan nuqtalardagi qiymatlari $f(P_1), f(P_2), ..., f(P_n)$ larni hisoblab ularni mos ravishda $\Delta S_1, \Delta S_2, ..., \Delta S_n$ yuzchalarga ko’paytirib

$$V_n = f(P_1)\Delta S_1 + f(P_2)\Delta S_2 + .. + f(P_n)\Delta S_n = \sum_{k=1}^{n} f(P_k)\Delta S_k \qquad (1.2)$$

yig’indini tuzamiz. Bu yig’indi D sohada $f(x, y)$ funksiya uchun **integral yig’indi** deb ataladi.

D sohada $f(x, y) \geq 0$ bo’lganda har bir $f(P_k)\Delta S_k$ ko’paytma geometrik jihatdan asosi ΔS_k, balandligi $f(P_k)$ bo’lgan **silindrcha (ustuncha) ning hajmini** ifodalaydi.

1-ta‘rif. (1.2) integral yig’indi $\Delta S_1, \Delta S_2, ..., \Delta S_n$ yuzchalarning diametrlarining eng kattasi 0 ga intilganda D sohani $\Delta S_1, \Delta S_2, ..., \Delta S_n$ yuzchalarga bo’linish usuli hamda bu yuzchalarda $P_1, P_2, ..., P_n$ nuqtalarning tanlanishiga bog'liq bo'lmagan aniq limitga ega bo'lsa, bu limit $f(x, y)$ funksiyadan D soha bo`yicha olingan **ikki o'lchovli integral** deyiladi va

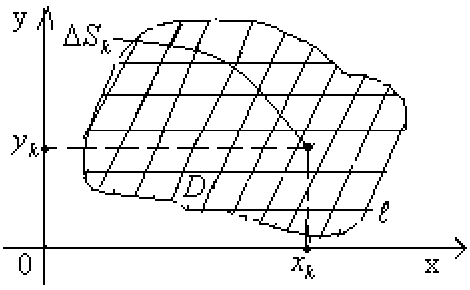
$$\iint_D f(P)dS \text{ yoki } \iint_D f(x, y)dxdy$$

ko'rinishda belgilanadi.

Demak, ta'rifga binoan

$$\lim_{\max diam \Delta S_k \to 0} \sum_{k=1}^{n} f(P_k)\Delta S_k = \iint_D f(x, y)dxdy .$$

Bu yerda D soha **integrallash sohasi** deyiladi.



2-chizma.

2-ta'rif. Chekli limit $\lim_{\max diam \Delta S_k \to 0} \sum_{k=1}^{n} f(P_k)\Delta S_k$ mavjud bo'lsa $f(x, y)$ funksiya D sohada **integrallanuvchi** deb ataladi.

Xuddi bir o'zgaruvchili funksiyadagidek $f(x, y)$ funksiyaning integrallanuvchi bo'lishi uchun uning chegaralangan bo'lishi zarur. Ammo bu shart yetarli emas. Masalan, $0 \le x \le 1,\ 0 \le y \le 1$ kvadratda aniqlangan

$$f(x, y) = \begin{cases} 1, agar\, x, y\ ratsional sonlar bo'lsa, \\ 0, agar\, x, y\ lardan kamida biri irratsional son bo'lsa \end{cases}$$

funksiya integrallanuvchi emas. Da'voning to'g'riligi bevosita ikki o'lchovli integralning ta'rifidan kelib chiqadi.

1.1-teorema. Chegaralangan yopiq D sohada uzluksiz $z = f(x, y)$ funksiya shu sohada integrallanuvchidir.

Faqatgina uzluksiz funksiyalar integrallanuvchi bo'lar ekan deb o'ylash noto'g'ri. Chegaralangan yopiq D sohada uzlukli funksiyalar orasida integrallanuvchi bo'lganlari ham, integrallanuvchi bo'lmaganlari ham mavjud.

1.3. Ikki o'lchovli integralning geometrik ma'nosi

Ikki o'lchovli integral tushunchasidan foydalanib (1.1) yig'indining limitini

$$v = \iint_D f(x, y)dxdy$$

ko'rinishda yozish mumkin.

Shunday qilib ikki o'lchovli integral geometrik jihatdan jismning **hajmini** ifodalar ekan. Bu ikki o'lchovli integralning **geometrik ma'nosidir**.

Agar D da $f(x, y) \equiv 1$ bo'lsa, u holda ikki o'lchovli integralning qiymati son jihatdan integrallash sohasining yuzi S ga teng bo'ladi:

$$S = \iint_D 1 \cdot dxdy = \lim_{\max diam \Delta S_k \to 0} \sum_{k=1}^{n} 1 \cdot \Delta S_k = \lim_{\max diam \Delta S_k \to 0} S = S .$$

1.4. Ikki o'lchovli integralning mexanik ma'nosi

$0xy$ tekislikda joylashgan yupqa tekis moddiy D plastinka berilgan bo'lsin. Shu plastinkaning ΔS yuzchasini hamda undagi $P(x, y)$ nuqtani qaraymiz.

Yuzchaning Δm massasini uning yuzi ΔS ga nisbati, ya'ni $\frac{\Delta m}{\Delta S}$ nisbat, ΔS yuzchaning o'rtacha **sirt zichligi** deb ataladi. Agar $\frac{\Delta m}{\Delta S}$ nisbat ΔS yuzcha $P(x, y)$ nuqtaga tortilganda γ limitga ega bo'lsa, bu limit D plastinkaning P nuqtadagi **sirt zichligi** deyiladi. Bu limit P nuqtaning vaziyatiga bog'liq bo'ladi. Shuning uchun u P nuqtaning koordinatalarini qandaydir funksiyasi bo'ladi: $\gamma = \gamma(x, y)$.

Sirt zichligi $\gamma = \gamma(x, y)$ $P(x, y)$ nuqtani koordinatalarining uzluksiz funksiyasi sifatida berilganda D plastinkaning massasi m ni aniqlaymiz. Agar D plastinka bir jinsli, ya'ni γ zichlik uning barcha nuqtalarida o'zgarmas $\gamma = \gamma_0$ bo'lsa, u holda plastinkaning massasi

$$m = \gamma_0 S \qquad (1.3)$$

bo'ladi, bunda S plastinkaning yuzi.

Sirt zichligi γ nuqtanining vaziyatiga bog'liq ravishda o'zgarganda (1.3) formula D plastinkani massasini aniqlash uchun yaramaydi.

D plastinkani ixtiyoriy chiziqlar bilan n ta kichik $\Delta S_1, \Delta S_2, ..., \Delta S_n$ yuzchalarga ajratib har bir ΔS_k yuzchada bittadan $P_k(x_k, y_k)$ nuqtani olamiz. Agar ΔS_k yuzcha juda kichik bo'lsa, u holda zichlik P_k nuqtadagi $\gamma_k = \gamma(x_k, y_k)$ zichlikdan juda kam farq qiladi. Har bir ΔS_k yuzchaning barcha nuqtalarida zichlik o'zgarmas va u taqriban tanlangan P_k nuqtadagi zichlikka teng deb hisoblab ΔS_k yuzchaning massasi Δm_k ni taqriban hisoblaymiz:

$$\Delta m_k \approx \gamma_k \Delta S_k = \gamma(x_k, x_k)\Delta S_k \ \left(k = \overline{1, n}\right).$$

D plastinkaning massasi $m = \sum_{k=1}^{n} \Delta m_k$ bo'lganligi sababli uni hisoblash uchun

$$m = \sum_{k=1}^{n} \Delta m_k \approx \sum_{k=1}^{n} \gamma(x_k, x_k)\Delta S_k$$

taqribiy tenglikka ega bo'lamiz.

Izlanayotgan massaning aniq qiymati sifatida $\sum_{k=1}^{n} \gamma(x_k, x_k)\Delta S_k$ yig'indini $\max diam \Delta S_k \to 0$ dagi limitni qabul qilamiz. Bu yig'indi D sohada uzluksiz $\gamma = \gamma(x, y)$ funksiya uchun integral yig'indi bo'lganligi sababli u $\max diam \Delta S_k \to 0$ da aniq chekli

$$\iint_D \gamma(x, y)dxdy$$

limitga ega bo'ladi.

Shunday qilib D plastinkaga joylashgan modda massasi sirt zichligidan D soha bo'yicha olingan ikki o'lchovli integralga teng bo'lar ekan:

$$m = \iint_D \gamma(x, y)dxdy$$

Ikki o'lchovli integralning **mexaniq ma'nosi** shundan iborat.

1.5. Ikki o'lchovli integralning xossalari

Ikki o‘zgaruvchili $f(x, y)$ funksiya uchun kiritilgan ikki o’lchovli integral tushunchasi bir o‘zgaruvchili $f(x)$ funksiya uchun kiritilgan aniq integral tushunchasining bevosita umumlashmasi bo’lgani uchun u aniq integralning hamma xossalariga ega. Shuning uchun ikki o’lchovli integralning xossalarini isbotsiz keltiramiz, bunda integral ostidagi funksiyalar qaralayotgan sohada integrallanuvchi deb faraz qilamiz..

1-xossa. O’zgarmas ko’paytuvchini ikki o’lchovli integral belgisidan tashqariga chiqarish mumkin, ya‘ni agar k -o’zgarmas son bo’lsa, u holda:

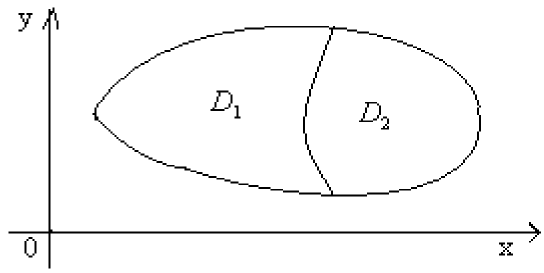
$$\iint_D kf(x,y)dxdy = k\iint_D f(x,y)dxdy.$$

2-xossa. Bir necha integrallanuvchi funksiyalarning algebraik yig’indisidan olingan ikki o’lchovli integral qo‘shiluvchilardan olingan ikki o’lchovli integrallarning algebraik yig’indisiga teng:

$$\iint_D [f(x,y) \pm g(x,y)]dxdy = \iint_D f(x,y)dxdy \pm \iint_D g(x,y)dxdy .$$

3-xossa. Agar D integrallash soha umumiy nuqtalarga ega bo’lmagan bir nechta qismlarga bo’linsa(3-chizma), u holda integrallanuvchi funksiyalardan butun D soha bo‘yicha olingan ikki o’lchovli integral har qaysi qismlar bo‘yicha olingan ikki o’lchovli integrallarning yig’indisiga teng:

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy .$$



3-chizma.

4-xossa. Agar D sohada integrallanuvchi $f(x, y)$ funksiya shu sohada $f(x, y)\geq 0$ bo’lsa, u holda $\iint_D f(x,y)dxdy \geq 0$ bo’ladi; agar D sohada $f(x, y)\leq 0$ bo’lsa, u holda $\iint_D f(x,y)dxdy \leq 0$ bo’ladi.

5-xossa. Agar D sohada $f(x, y)$, $g(x, y)$ funksiyalar integrallanuvchi va $f(x, y)\geq g(x, y)$ bo’lsa, u holda:

$$\iint_D f(x,y)dxdy \geq \iint_D g(x,y)dxdy .$$

6-xossa (o’rta qiymat haqida). Agar $f(x, y)$ funksiya yopiq chegaralangan D sohada **uzluksiz** bo’lsa, u holda bu sohada shunday $P(x_0, y_0)$ nuqta mavjud bo’lib

$$\iint_D f(x,y)dxdy = f(x_0,\ y_0)\cdot S .$$

tenglik bajariladi, bu yerdagi S-D sohaning yuzi. Funksiyaning

$$f(x_0,\ y_0) = \frac{1}{S}\iint_D f(x,y)dxdy$$

qiymati $f(x, y)$ funksiyaning D sohadagi **o’rta qiymati** deyiladi.

7-xossa (integralning chegaralanganligi haqida). Agar $f(x, y)$ funksiya yopiq chegaralangan D sohada uzluksiz hamda M va m uning shu sohadagi **eng katta** va **eng kichik** qiymatlari bo’lsa,

$$mS \le \iint_D f(x,y)dxdy \le M\,S$$

tengsizliklar o’rinli, bu yerdagi $S - D$ sohaning yuzi.

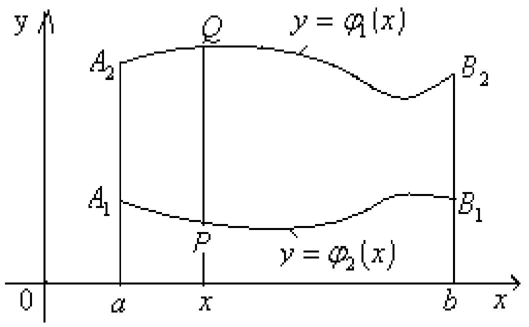
1.6.Ikki o’lchovli integralni hisoblash

Aniq integral tushunchasi kiritilganda aniq integralni uni ta’rifiga asoslanib integral yig‘indining limiti kabi hisoblash katta qiyinchiliklarga olib kelishi ta’kidlangan edi.

Shuningdek ikki o‘lchovli integralni ham integral yig‘indining limiti sifatida hisoblash nihoyatda uzundan-uzoq hisoblashlarni talab qilishi mumkin. Odatda ikki o‘lchovli integralni hisoblash ikkita aniq integralni ketma-ket hisoblashga keltiriladi. Bu qanday qilinishini ko‘rsatamiz. Osonlik uchun integrallash sohasi D da integral ostidagi funksiya $f(x, y) \ge 0$ deb faraz qilamiz. Bunaqa faraz ikki o‘lchovli integralni silindrik jismning hajmi deb qarash imkonini beradi.

$\iint_D f(x,y)dxdy$ ikki o‘lchovli integralni hisoblash talab etilsin, bunda $f(x, y)$ funksiya D sohada uzluksiz.

Integrallash sohasi D $y = \varphi_1(x), y = \varphi_2(x)$ uzluksiz funksiyalarning grafiklari hamda $x = a,\ x = b$ vertikal to‘g‘ri chiziqlar bilan chegaralangan va a bilan b orasidagi barcha x lar uchun $\varphi_2(x) \ge \varphi_1(x)$ tengsizlik bajariladi deb faraz qilamiz (4-chizma).



4-chizma.

$0x$ o‘qning $(x,0)$ $(a < x < b)$ nuqtasi orqali $0y$ o‘qqa parallel to‘g‘ri chiziq o‘tkazamiz.

Bu to‘g‘ri chiziq D sohaning chegarasini ikkita $P(x, \varphi_1(x))$ va $Q(x, \varphi_2(x))$ nuqtalarda kesib o‘tadi. P nuqtani **kirish nuqtasi** Q nuqtani esa **chiqish nuqtasi** deb ataymiz.

Shuningdek A_1PB_1 egri chiziqni kirish, A_2QB_2 egri chiziqni chiqish chegarasi deymiz.

Ikki o‘lchovli integralni hisoblashga kirishishdan oldin integrallash sohasining ba’zi turlari bilan tanishamiz.

1-ta’rif. Agar:

a) D sohaning ichki nuqtasidan $0y$ o‘qqa parallel o‘tuvchi har qanday to‘g‘ri chiziq sohaning chegarasini ikki nuqtada kesib o‘tsa;

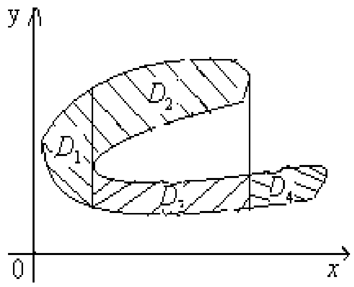
b) kirish va chiqish chegaralarining har biri alohida tenglamalar bilan berilsa, D soha $0y$ o‘q yo‘nalishida **muntazam (to‘g‘ri) soha** deyiladi.

$0x$ o‘q yo‘nalishida **muntazam (to‘g‘ri) soha** ham shunga o‘xshash aniqlanadi.

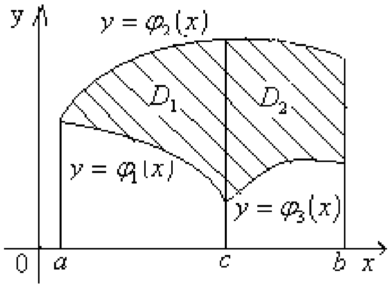
Ham $0x$, ham $0y$ o‘qlari yo‘nalishida muntazam (to‘g‘ri) bo‘lgan sohani qisqacha **muntazam (to‘g‘ri) soha** deymiz.

Agar ta’rifdagi shartlardan aqalli bittasi buzilsa, u holda D soha u yoki bu yo‘nalishda **nomuntazam (noto‘g‘ri) soha** deyiladi. Bunday holda sohani $0y$ yoki $0x$ o‘qiga parallel to‘g‘ri chiziqlar bilan har biri u yoki bu o‘q yo‘nalishiga nisbatan muntazam bo‘ladigan qismlarga ajratish mumkin.

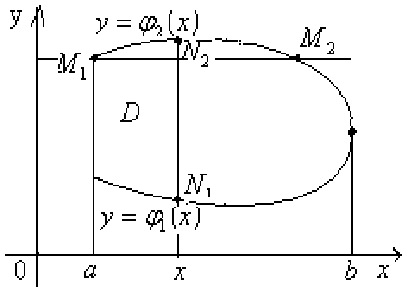
5-chizmada $0y$ o‘q yo‘nalishida nomuntazam soha tasvirlanganki, bunda a) shart buziladi: bunda soha chegarasini to‘rtta nuqtada kesadigan $0y$ o‘qqa parallel to‘g‘ri chiziq mavjud. Bu sohani $0y$ o‘qqa parallel to‘g‘ri chiziqlar bilan to‘rtta D_1, D_2, D_3, D_4 $0y$ o‘q yo‘nalishida muntazam sohalarga bo‘lish mumkin.



5-chizma.



6-chizma.



7-chizma.

6-chizmada $0y$ o‘q yo‘nalishida nomuntazam soha tasvirlanganki, bunda b) shart buzilgan: kirish chegarasi ikkita tenglama bilan berilgan. Bu sohani $x = c$ to‘g‘ri chiziq bilan ikkita $0y$ o‘q yo‘nalishida muntazam D_1 va D_2 sohaga bo‘lish mumkin.

127-chizmada $0y$ o‘q yo‘nalishida **muntazam (to‘g‘ri) soha** tasvirlangan.

Endi ikki o‘lchovli

$$\iint_D f(x, y)dxdy$$

integralga qaytamiz. Farazimizga ko‘ra integrallash sohasi D $0y$ o‘q yo‘nalishida muntazam soha va $f(x, y) \geq 0$. Qilingan faraz ikki o‘lchovli integralni yuqoridan $z = f(x, y)$ sirtning D sohaga proyeksiyalanadigan qismi bilan, quyidan D soha bilan hamda yo‘naltiruvchisi D sohaning chegarasidan iborat, yasovchisi $0z$ o‘qqa parallel silindrik jismning hajmiga tengligidan foydalanish imkonini beradi, ya’ni

$$V = \iint_D f(x, y)dxdy$$

tenglikdan foydalanish imkonini beradi.

Endi silindrik jismning V hajmini o‘zimizga tanish ko‘ndalang kesimlar usulidan foydalanib hisoblaymiz (8-chizma).

Ma’lumki, jismning $0x$ o‘qqa perpendikulyar $x = x$ $\quad a \leq x \leq b$ tekislik bilan kesganda hosil bo‘lgan kesimning yuzi $S(x)$ ma’lum bo‘lsa, u holda bu jismning hajmi

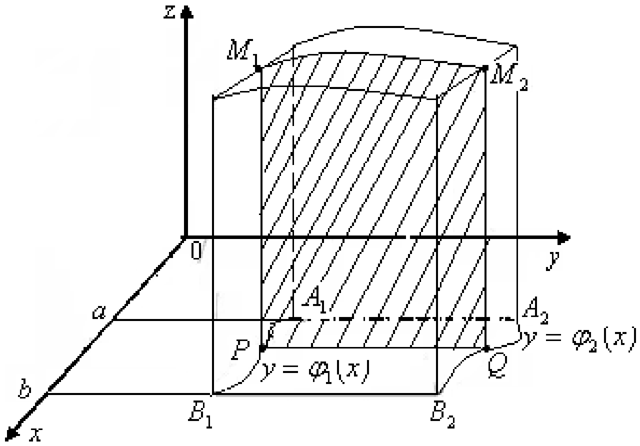
$$V = \int_a^b S(x)\,dx \qquad (1.4)$$

formula yordamida topiladi. Silindrik jismning hajmini shu formuladan foydalanib hisoblaymiz. $(x;0;0)$ nuqta orqali $0x$ o‘qqa perpendikulyar tekislik o‘tkazib kesimda PM_1M_2Q egri chiziqli trapetsiyani hosil qilamiz (8-chizma).

M_1M_2 egri chiziq nuqtalarining applikatasi $z = f(x, y)$ x o‘zgarmas bo‘lganligi uchun faqat y ning funksiyasi bo‘lib, bunda y $\varphi_1(x)$ dan $\varphi_2(x)$ gacha o‘zgaradi. PM_1M_2Q egri chiziqli trapetsiyaning $S(x)$ yuzi

$$S(x)=\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \qquad (1.5)$$

bo‘lishi ravshan. Shunday qilib, silindrik jismning ko‘ndalang kesimi yuzi (1.5) formula yordamida topilar ekan.



8-chizma.

(1.4) formulaga $S(x)$ ning qiymatini qo‘yib

$$V=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right)dx \qquad (1.6)$$

ga ega bo‘lamiz. Ikkinchi tomondan silindrik jismning hajmi $V=\iint\limits_D f(x,y)dxdy$ ikki o‘lchovli integralga teng bo‘lganligi sababli

$$\iint\limits_D f(x,y)dxdy=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right)dx$$

yoki

$$\iint\limits_D f(x,y)dxdy=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \qquad (1.7)$$

bo‘ladi. Bu ikki o‘lchovli integralni hisoblash formulasidir.

(1.7) formulaning o‘ng tomonidagi ifoda takroriy yoki **ikki karrali integral** deyiladi, shu bilan birga $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$ **ichki integral** deb ataladi. Bu aniq integralni hisoblash jarayonida x **o‘zgarmas** sanaladi.

Ikki o‘lchovli integralni hisoblash uchun avval ichki integral hisoblanadi. Hisoblash natijasida x ning qandaydir funksiyasi (yoki o‘zgarmas son) ga ega bo‘lamiz Bu funksiya tashqi integral uchun integral ostidagi funksiya bo‘ladi. Bu funksiyani a dan b gacha chegarada integrallasak qaralayotgan ikki o‘lchovli integralning qiymati kelib chiqadi.

(1.7) formula D sohada na faqat $f(x, y)\geq 0$ bo‘lgandagina balki $f(x, y)\leq 0$ bo‘lganda ham yoki $f(x, y)$ funksiya D sohada ishorasini o‘zgartirganda ham to‘g‘riligicha qoladi.

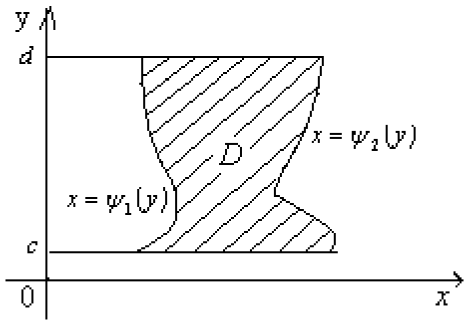
1-eslatma. $0x$ o‘q yo‘nalishida muntazam D soha $y=c$, $y=d$ $(c<d)$ gorizontal to‘g‘ri chiziqlar hamda $[c,d]$ kesmada uzluksiz $x=\psi_1(y)$, $x=\psi_2(y)$, $(\psi_1(y)\le\psi_2(y))$ funksiyalarning grafiklari bilan chegaralanganda (9-chizma)

$$\iint\limits_D f(x,y)dxdy=\int\limits_c^d\left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right)dy$$

yoki

$$\iint\limits_D f(x,y)dxdy=\int\limits_c^d dy\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \qquad (1.8)$$

tengliklar o‘rinliligini ko‘rsatish mumkin. Bunda ichki inetgralda integrallash x bo‘yicha amalga oshirilib, integrallash jarayonida y o‘zgarmas sanaladi. Integrallash natijasida y ning qandaydir funksiyasi (yoki o‘zgarmas son) ga ega bo‘lamiz. Uni c dan d gacha chegarada integrallasak ikki o‘lchovli integralning qiymati hosil bo‘ladi.



9-chizma.

2-eslatma. Tashqi integralning chegaralari doimo o‘zgarmas son bo‘ladi.

3-eslatma. Agar D integrallash sohasi nomuntazam bo‘lsa, uni bir necha $0y$ yoki $0x$ yo‘nalishdagi muntazam sohalarga bo‘lish, bu muntazam sohalarning har biri bo‘yicha ikki o‘lchovli integrallarni hisoblash va shundan keyin natijalarni jamlash kerak. Ikki o‘lchovli integralning 3-xossasiga ko‘ra D soha bo‘yicha olingan ikki o‘lchovli integral shu yig‘indiga teng bo‘ladi. Masalan, 5-chizmada tasvirlangan nomuntazam soha bo‘yicha ikki o‘lchovli integral uchun

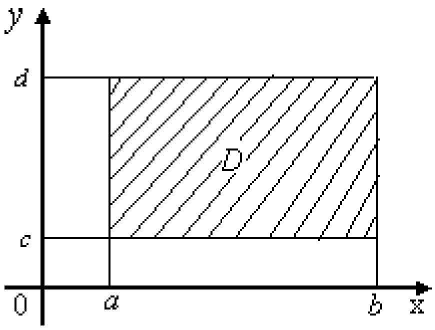
$$\iint\limits_D f(x,y)dxdy=\iint\limits_{D_1} f(x,y)dxdy+\iint\limits_{D_2} f(x,y)dxdy+\iint\limits_{D_3} f(x,y)dxdy+\iint\limits_{D_4} f(x,y)dxdy$$

tenglik o‘rinlidir. Bu tenglikning o‘ng tomonidagi barcha integrallar $0y$ yoki $0x$ yo‘nalishdagi muntazam sohalarga bo‘yicha olingandir.

4-eslatma. Agar integrallash sohasi D

$$a\le x\le b,\ c\le y\le d$$

to‘g‘ri to‘rtburchakdan (10-chizma) iborat bo‘lsa, u holda (1.7) va (1.8) formulalar quyidagi ko‘rinishni oladi:



10-chizma.

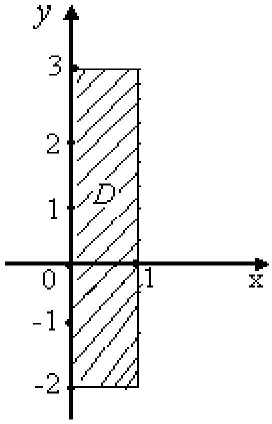
$$\iint\limits_D f(x,y)dxdy=\int\limits_a^b dx\int\limits_c^d f(x,y)dy, \qquad (1.9)$$

$$\iint\limits_D f(x,y)dxdy=\int\limits_c^d dy\int\limits_a^b f(x,y)dx. \qquad (1.10)$$

Shunday qilib, ikki o‘lchovli integral ikki karraliga keltirilib hisoblanar ekan. Ba’zan ikki karrali integral $\int\limits_a^b\int\limits_c^d f(x,y)dydx$ ko‘rinishda yoziladi. Bunday holda ichki integral, differensiali birinchi o‘rinda turgan o‘zgaruvchi bo‘yicha amalga oshiriladi.

Endi ikki o‘lchovli integrallarni hisoblashga doir misollar qaraymiz.

1-misol. $I=\iint\limits_D xy^2dxdy$ ikki o’lchovli integral hisoblansin,



11-chizma.

bunda D-to‘g‘ri to’rt burchak 0≤x≤1, -2≤y≤3.

Yechilishi. Integrallash chegaralarini qoyib berilgan ikki o’lchovli integralni ikki karrali integralga keltirib hisoblaymiz:

$$I=\int_0^1\left(\int_{-2}^3 xy^2dy\right)dx=\int_0^1 x\left(\int_{-2}^3 y^2dy\right)dx=\int_0^1 x\left(\frac{y^3}{3}\bigg|_{-2}^3\right)dx=\int_0^1 x\left(\frac{3^3-(-2)^3}{3}\right)dx$$

$$=$$

$$=\frac{35}{3}\int_0^1 xdx=\frac{35}{3}\,\frac{x^2}{2}\bigg|_0^1=\frac{35}{3}\,\frac{1}{2}=\frac{35}{6}.$$

2-misol. Tomoni $2a$ ga teng kvadrat shakldagi plastinkaning istalgan $M(x,\,y)$ nuqtasidagi $\gamma(x,\,y)$ zichligi shu nuqtadan kvadratning diagonallari kesishish nuqtasigacha masofaning kvadratiga proporsional bo‘lib, k proporsionallik koeffitsiyenti bo‘lganda plastinkaning massasi m topilsin.

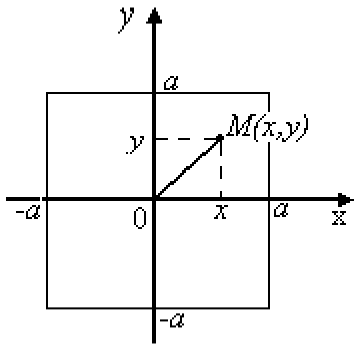
Yechilishi. Koordinatalar sistemasini 12-chizmada ko‘rsatilgandek tanlaymiz. $M(x,\,y)$ plastinkaning ixtiyoriy nuqtasi bo‘lsin. U holda $M(x,\,y)$ nuqtadan diagonallarning kesishish nuqtasigacha masofaning kvadrati x^2+y^2 ga teng bo‘ladi. Natijada, M nuqtadagi sirt zichligi

$$\gamma(M)=\gamma(x,\,y)=k\left(x^2+y^2\right)$$

bo‘ladi. Massani topish formulasi $m=\iint\limits_D \gamma(x,y)dxdy$ ga binoan

$$m=\iint\limits_D k(x^2+y^2)dxdy$$

bo‘ladi.



12-chizma.

Integral ostidagi funksiya x va y ga nisbatan juft, integrallash sohasi koordinatalar o‘qlariga nisbatan simmetrikligini hisobga olib plastinkaning birinchi chorakda joylashgan qismining massasini topib uni to‘rtga ko‘paytirsak plastinkaning massasi aniqlanadi:

$$m=4k\int_0^a dx\int_0^a\left(x^2+y^2\right)dy=4k\int_0^a\left(x^2y+\frac{y^3}{3}\bigg|_0^a\right)dx=4k\int_0^a\left(x^2a+\frac{a^3}{3}\right)dx=$$

$$=4k\left(\frac{ax^3}{3}+\frac{a^3}{3}x\bigg|_0^a\right)=4k\cdot\frac{2a^4}{3}=\frac{8}{3}ka^4.$$

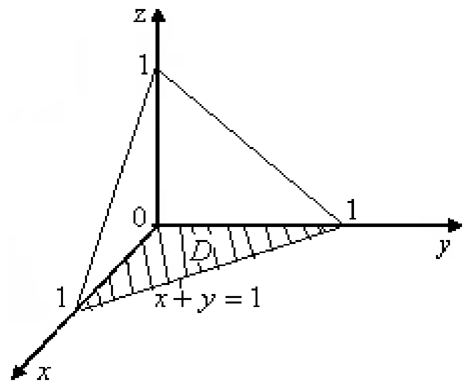
.

3-misol. $x=0, y=0, z=0$ va $x+y+z=1$ (13-chizma) sirtlar bilan chegaralangan jismning hajmi topilsin

Yechilishi. Hajmni topish formulasi $V=\iint\limits_D f(x,y)dxdy$ ga binoan

$$V=\iint\limits_D (1-x-y)dxdy$$

ga ega bo‘lamiz. Bu yerdagi integrallash sohasi D $x=0, y=0, x+y=1$ to‘g‘ri chiziqlar bilan chegaralangan uchburchak. Oxirgi ikki o’lchovli integralga integrallash chegaralarini qo‘yib quyidagiga ega bo‘lamiz:



13-chizma.

$$V=\int_0^1 dx\int_0^{1-x}(1-x-y)dy=\int_0^1\left[(1-x)y-\frac{y^2}{2}\right]\Bigg|_0^{1-x}dx=\int_0^1\left[(1-x)^2-\frac{(1-x)^2}{2}\right]dx=$$

$$=\frac{1}{2}\int_0^1(1-x)^2dx=-\frac{1}{2}\int_0^1(1-x)^2d(1-x)=-\frac{1}{2}\cdot\frac{(1-x)^3}{3}\Big|_0^1=\frac{1}{6}.$$

4-misol. $y^2=x+1$ va $x+y=1$ chiziqlar bilan chegaralangan D sohaning yuzi topilsin (14-chizma).

Yechilishi. D soha chapdan $y^2=x+1$ parabola, o‘ngdan $x+y=1$ to‘g‘ri chiziqlar bilan chegaralangan.

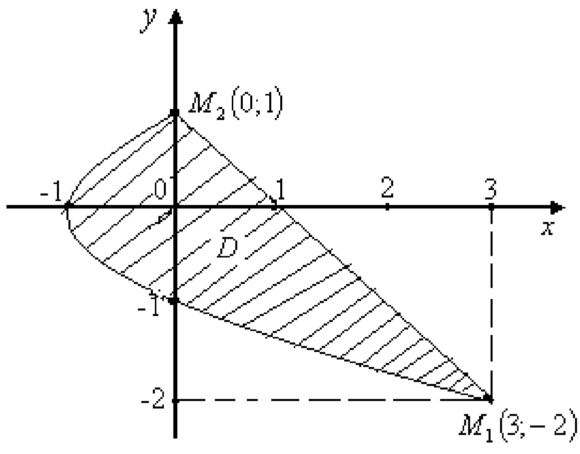
$$\begin{cases} y^2=x+1, \\ y=x+1 \end{cases}$$

sistemani yechib parabola bilan to‘g‘ri chiziqning kesishish nuqtalari $M_1(3;-2)$ va $M_2(0;1)$ ni aniqlaymiz. D soha $0x$ o‘q yo‘nalishida muntazam bo‘lganligi uchun ichki integralda

integrallash o‘zgaruvchisi sifatida x ni olamiz. Chiziqlarni tenglamalarini x ga nisbatan yechsak $x=y^2-1,\ x=1-y$ bo‘ladi. Demak $0x$ o‘q yo‘nalishida muntazam D soha chapdan $x=y^2-1$ parabola, o‘ngdan $x=1-y$ to‘g‘ri chiziq bilan chegaralanganligi uchun sohaning yuzini topish formulasi

$$S=\iint\limits_D dxdy$$

ga integrallash chegarasini qo‘yib quyidagiga ega bo‘lamiz.



14-chizma.

$$S=\int_{-2}^{1}dy\int_{y^2-1}^{1-y}dx=\int_{-2}^{1}\left(x\Big|_{y^2-1}^{1-y}\right)dy=\int_{-2}^{1}(2-y-y^2)dy=\left(2y-\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_{-2}^{1}=$$

$$=\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(2\cdot(-2)-\frac{(-2)^2}{2}-\frac{(-2)^3}{3}\right)=\frac{7}{6}+\frac{10}{3}=\frac{27}{6}=\frac{9}{2}.$$

O'z-o'zini tekshirish uchun savollar

1. Qanaqa soha yopiq soha deyiladi?
2. Soha qachon chegaralangan deyiladi?
3. Funksiyaning berilgan soha bo'yicha ikki o'lchovli integrali deb nimaga aytiladi?
4. Ikki o'lchovli integralning mavjudligi haqidagi teorema nimadan iborat.
5. Ikki o'lchovli integralning geometrik va mexanik ma'nolarini tushuntiring.
6. Ikki o'lchovli integral yordamida yuza, hajm va massani topish formulalarini yozing.
7. Ikki o'lchovli integralning xossalarini ayting.
8. Ikki o'lchovli integral uchun o'rta qiymat haqidagi teoremani va integralning chegaralanganligi haqidagi teoremalarni ifodalang.
9. Muntazam (to'g'ri) sohaga ta'rif bering.
10. Ikki karrali integral deb nimaga aytiladi?
11. Ikki o'lchovli integral qanday hisoblanadi?

Mustaqil yechish uchun mashqlar va test savollari

Quyidagi chiziqlar bilan chegarlangan yuzlar ikki o'lchovli integrallar bilan yozilsin va hisoblansin:

1. $y^2 = 4ax,\ x + y = 3a,\ y = 0$. Javob: $\frac{10a^2}{3}$.

2. $y = \sin x,\ y = \cos x,\ x = 0$. Javob: $\sqrt{2} - 1$.

3. $y^2 = 2x,\ y = x$. Javob: $\frac{2}{3}$.

4. $y = -2,\ y = x + 2,\ y = 2,\ y^2 = x$

A) $\frac{40}{3}$ B) $\frac{40}{5}$ C) 15 D) 16.

5. $y = \frac{1}{a}(x - a)^2$ $(a. > 0);\ x^2 + y^2 = a^2$

A) $\frac{a^2}{12}(3\pi - 4)$ B) $\frac{a^2}{12}(3\pi - 5)$ C) $\frac{a^2}{6}(3\pi - 4)$ D) $\frac{a^2}{9}(3\pi - 4)$.

6. $y = x,\ y = 5x,\ x = 1$. Javob: $\frac{1}{2}$.

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Javob: πab.

8. $xy = 4,\ y = x,\ x = 4$. Javob: $6 - 4\ell n2 \approx 3{,}28$.

9. $y = x^2,\ 4y = x^2,\ y = 4$. Javob: $10\frac{2}{3}$ kv.birl.

10. $y = x^2,\ 4y = x^2,\ y = \pm 2$. Javob: 4 kv.birl.

11. $y^2 = 4 + x,\ x + 3y = 0$. Javob: $20\frac{5}{6}$.

12. $ay = x^2 - 2ax,\ y = x$. Javob: $\frac{9a^2}{2}$.

13. $y = \ell nx,\ x - y = 1,\ y = -1$. Javob: $\frac{1}{2} - \frac{1}{e}$.

14. $r = a(1 - \cos\varphi),\ r = a$ chiziqlar bilan chegaralanib, doira tashqarisida joylashgan soha yuzi hisoblansin. Javob: $\left(\frac{\pi}{4} + 2\right)a^2$.

15. $r\cos\varphi = a$ to‘g‘ri chiziq va $r = 2a$ aylana bilan chegaralangan yuz hisoblansin. Javob: $\left(\frac{4\pi}{3} - \sqrt{3}\right)a^2 \approx 2{,}457a^2$.

Integrallash sohasi D qavs ichida ko‘rsatilgan to‘g‘ri to‘rtburchakdan iborat bo‘lganda quyidagi ikki o‘lchovli integrallar hisoblansin;

16. $\iint\limits_D xydxdy \qquad (0 \le x \le 1,\ \ 0 \le y \le 2)$. Javob: 1.

17. $\iint\limits_D e^{x+y}dxdy \qquad (0 \le x \le 1,\ \ 0 \le y \le 1)$. Javob: $(e-1)^2$.

18. $\iint\limits_D \frac{x^2}{1+y^2}dxdy \qquad (0 \le x \le 1,\ \ 0 \le y \le 1)$. Javob: $\frac{\pi}{12}$.

19. $\iint\limits_D \frac{dxdy}{(x+y+1)^2} \qquad (0 \le x \le 1,\ \ 0 \le y \le 1)$. Javob: $\ell n\frac{4}{3}$.

20. $\iint\limits_D \frac{ydxdy}{\left(1+x^2+y^2\right)^{\frac{3}{2}}} \qquad (0 \le x \le 1,\ \ 0 \le y \le 1)$. Javob: $\ell n\frac{2+\sqrt{2}}{1+\sqrt{3}}$.

$\iint\limits_D f(x,y)dxdy$ ikki o‘lchovli integral ikki karrali integral ko‘rinishida yozilsin.

21. D soha $x = 3,\ x = 5,\ 3x - 2y + 4 = 0,\ 3x - 2y + 1 = 0$ to‘g‘ri chiziqlar bilan chegaralangan parallelogramm. Javob: $\int\limits_3^5\left(\int\limits_{\frac{3x+1}{2}}^{\frac{3x+4}{2}} f(x,y)\,dy\right)dx$.

22. D soha $x = 0,\ y = 0,\ x + y = 2$ chiziqlar bilan chegaralangan uchburchak. Javob: $\int\limits_0^2\left(\int\limits_0^{2-x} f(x,y)\,dy\right)dx$.

23. D soha: $x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0$. Javob: $\int\limits_0^1\left(\int\limits_0^{\sqrt{1-x^2}} f(x,y)\,dy\right)dx$.

24. D soha: $y \ge x^2,\ y \le 4 - x^2$. Javob: $\int\limits_{-\sqrt{2}}^{\sqrt{2}}\left(\int\limits_{x^2}^{4-x^2} f(x,y)\,dy\right)dx$.

25. D soha: $(x-2)^2 + (y-3)^2 \le 4$. Javob: $\int\limits_0^4\left(\int\limits_{3-\sqrt{4x-x^2}}^{3+\sqrt{4x-x^2}^{\,2}} f(x,y)\,dy\right)dx$.

Integrallash tartibi o‘zgartirilsin.

26. $\int_0^1\left(\int_y^{\sqrt{y}} f(x,y)\,dx\right)dy$. Javob: $\int_0^1\left(\int_{x^2}^{x} f(x,y)\,dy\right)dx$.

27. $\int_{-1}^1\left(\int_{\frac{1}{\sqrt{2}}\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\right)dx$. Javob: $\int_0^1\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\right)dy$.

28. $\int_0^r\left(\int_x^{\sqrt{2rx-x^2}} f(x,y)\,dy\right)dx$. Javob: $\int_0^r\left(\int_{r-\sqrt{r^2-y^2}}^{y} f(x,y)\,dx\right)dy$.

Berilgan integrallar hisoblansin.

29. $\int_0^a\left(\int_0^{\sqrt{x}} dy\right)dx$. Javob: $\frac{2}{3}a^{\frac{3}{2}}$.

30. $\int_2^4\left(\int_x^{2x} \frac{y}{x}dy\right)dx$. Javob: 9.

31. $\int_1^2\left(\int_0^{\ln y} e^y dx\right)dy$. Javob: $\frac{1}{2}$.

32. $\iint_D x^3y^2dxdy,$ $D-$ doira $x^2+y^2\le R^2$. Javob: 0.

33. $\iint_D (x^2+y)dxdy,$ $D-y=x^2$ va $y^2=x$ parabolalar bilan chegaralangan soha. Javob: $\frac{33}{140}$.

34. $\iint_D \frac{x^2}{y^2}dxdy,$ $D-x=2,$ $y=x$ to‘g‘ri chiziqlar va $xy=1$ giperbola bilan chegaralangan soha. Javob: $\frac{9}{4}$.

35. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ $x=0,$ $y=0,$ $z=0$ tekisliklar bilan chegaralangan jismning hajmi topilsin. Javob: $\frac{abc}{6}$.

36. $x=0,$ $y=0,$ $z=0,$ $2x+3y-12=0$ tekisliklar va $z=\frac{1}{2}y^2$ sirt bilan chegaralangan jismning hajmi topilsin. Javob: 16 kub.birl.

37. $z=4x^2+2y^2+1,$ $x+y-3=0,$ $x=0,$ $y=0,$ $z=0$ sirtlar bilan chegaralangan jismning hajmi topilsin.
A) 45 B) 46 C) 47 D) 50.

38. $f(x,y)=1+x+y,$ $x=\sqrt{y},$ $y=-x,$ $y=2,$ $z=0$ sirtlar bilan chegaralangan jismning hajmi topilsin. Javob: $\frac{44}{15}\sqrt{2}+\frac{13}{3}$.

39. $z=x^2+y^2$ paraboloid, koordinata tekisliklari va $x+y=1$ tekislik bilan chegaralangan jismning hajmi topilsin. Javob: $\frac{1}{6}$.

40. $z = x^2 + y^2$ paraboloid va $z = 0,\ y = 1,\ y = 2x,\ y = 6 - x$ tekisliklar bilan chegaralangan jismning hajmi topilsin. Javob: $78\frac{15}{32}$.

41. $y = \sqrt{x},\ y = 2\sqrt{x}$ silindrlar hamda $z = 0$ va $x + z = 6$ tekisliklar bilan chegaralangan jismning hajmi topilsin. Javob: $\frac{48}{5}\sqrt{6}$.

42. $z = 9 - y^2$ silindr, koordinata tekisliklari va $3x + 4y = 12\ (y \geq 0)$ tekislik bilan chegaralangan jismning hajmi topilsin. Javob: 45.

43. $2y^2 = x$ silindr, $\frac{x}{4} + \frac{y}{2} + \frac{z}{4} = 1$ va $z = 0$ tekisliklar bilan chegaralangan jismning hajmi topilsin. Javob: $16\frac{1}{5}$.

44. $z = x^2 + y^2$ paraboloid, $y = x^2$ silindr va $y = 1,\ z = 0$ tekisliklar bilan chegaralangan jismning hajmi topilsin. Javob: $\frac{88}{105}$.

1-ma'ruza.давоми Mavzu: Ikki o'lchovli integralni integrallash usuli va tadbiqlari

Reja:

1. Ikki o'lchovli integralda o'zgaruvchilarni almashtirish.
2. Sirtning yuzini topish.
3. Ikki o'lchovli integralning mexanikaga tadbiqlari.

Adabiyotlar: 3,4,11,14,16,26.

Tayanch iboralar: yakobian, to'g'ri soha, noto'g'ri soha, sirtning yuzi, ikki o'lchovli integral, sirt zichligi, massa, statik moment, inersiya momenti, og'irlik markazi.

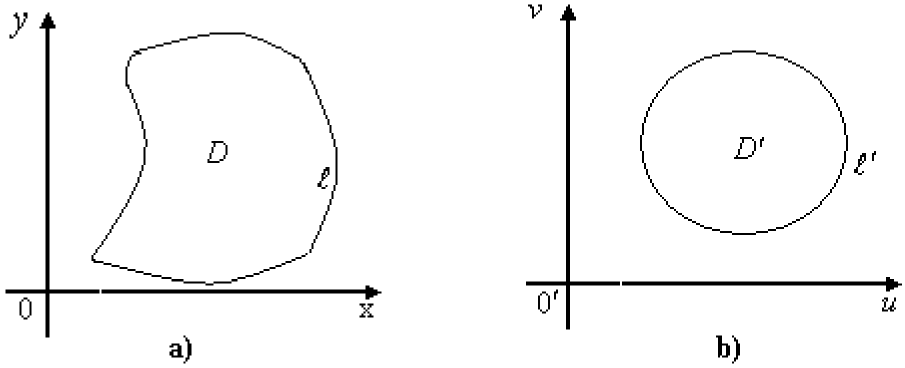
1.1. Ikki o'lchovli integralda o'zgaruvchilarni almashtirish

Aniq integralni hisoblashda o'zgaruvchini almashtirish muhim o'rin tutishi bizga ma'lum. Bu usul yordamida berilgan integralni jadval integraliga keltirilib, **Nyuton-Leybnis** formulasidan foydalanilar edi. Ikki o'lchovli integral uchun shunday usulni qaraymiz. $0xy$ tekislikda yopiq ℓ chiziq bilan chegaralangan D sohani hamda unda aniqlangan va uzluksiz $z = f(x, y)$ funksiyani qaraymiz.

$$\iint_D f(x, y)dxdy \qquad (1.1)$$

ikki o'lchovli integralni hisoblash talab etilsin. Bunda D shunday sohaki berilgan ikki o'lchovli integralni to'g'ridan-to'g'ri ikki karrali integral orqali ifodalash ancha ko'p hisoblashlarni bajarishni talab qiladi deb faraz qilamiz.

Integralda $x = \varphi(u, v),\ y = \psi(u, v)$ (1.2) formulalar yordamida, yangi u, v o'zgaruvchilarga o'tamiz, bunda $\varphi(u, v), \psi(u, v)$ funksiyalar D' sohada bir qiymatli, uzluksiz va uzluksiz xususiy hosilalarga ega bo’lgan funksiyalar. (1.2) formulalar D va D' sohalarning nuqtalari orasida o’zaro bir qiymatli moslik o’rnatadi, ya‘ni D sohadan olingan har bir $P(x,y)$ nuqtaga D' sohaning aniq $P'(u,v)$ nuqtasi va aksincha, D' sohaning har bir $P'(u,v)$ nuqtasiga D sohanining aniq bir $P(x,y)$ nuqtasi mos keladi deb faraz qilamiz. Bu holda $0xy$ tekislikdagi biror nuqta harakatlanib, D sohani chegaralovchi ℓ yopiq chiziqni chizganda $0'uv$ tekislikdagi shu nuqtaga mos nuqta harakatlanib, o’z navbatida biror D' sohani chegaralovchi yopiq ℓ' chiziqni chizadi (15-chizma).



15-chizma.

Agar $I=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\neq 0$ bo’lsa, u holda (1.1) integral uchun

$$\iint_D f(x,y)dxdy=\iint_{D'} f(\varphi(u,v),\psi(u,v))|I|dudv \qquad (1.3)$$

o’zgaruvchilarni almashtirish formulasi o‘rinli.

I determinant nemis matematigi Yakobi sharafiga **yakobian** deb ataladi.

Xususiy holda Dekart koorinatalaridan qutb koordinatalariga o’tilganda $u=\theta, v=\rho, x=\rho\cos\theta, y=\rho\sin\theta$ bo’lganligi sababli

$$I=\begin{vmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\rho}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\rho}\end{vmatrix}=\begin{vmatrix}(\rho\cos\theta)'_\theta & (\rho\cos\theta)'_\rho\\ (\rho\sin\theta)'_\theta & (\rho\sin\theta)'_\rho\end{vmatrix}=\begin{vmatrix}-\rho\sin\theta & \cos\theta\\ \rho\cos\theta & \sin\theta\end{vmatrix}=-\rho\sin^2\theta-\rho\cos^2\theta=-\rho,\ |I|=\rho$$

bo’lib (1.3) formula

$$\iint_D f(x,y)dxdy=\iint_{D'} f(\rho\cos\theta,\rho\sin\theta)\rho d\theta d\rho \qquad (1.4)$$

ko’rinishga ega bo’ladi.

1-misol. $y=x+1, y=x-3, y=-\frac{1}{3}x+\frac{7}{3}, y=-\frac{1}{3}x+5$ to`g`ri chiziqlar bilan chegaralangan D soha boyicha olingan $\iint_D (y-x)dxdy$ ikki o’lchovli integral hisoblansin.

Yechilishi. O’zgaruvchilarni almashtirish yordamida berilgan integralni tomonlari koordinata o’qlariga parallael bo’lgan to’g’ri to’rtburchak bo`yicha olingan integralga keltiramiz: $u = y - x, v = y + \frac{1}{3}x$ almashtirish olamiz. U vaqtda $y = x+1, y = x-3$ to'g'ri chiziqlar mos ravishda $0'uv$ tekislikdagi $u=1, u=-3$ to'g'ri chiziqlarga; $y = -\frac{1}{3}x + \frac{7}{3}, y = -\frac{1}{3}x + 5$ to'g'ri chiziqlar esa $v = \frac{7}{3}, v = 5$ to'g'ri chiziqlarga almashadi.

$\begin{cases} u = y - x, \\ v = y + \frac{1}{3}x \end{cases}$ sistemani yechib, quyidagilarni hosil qilamiz:

$x = -\frac{3}{4}u + \frac{3}{4}$, $y = \frac{1}{4}u + \frac{3}{4}v$. Bu tenglamalar yordamida $0xy$ tekislikdagi D parallelogramm $0'uv$ tekislikdagi D' to‘g‘ri to‘rtburchagiga akslanadi (16-chizma).

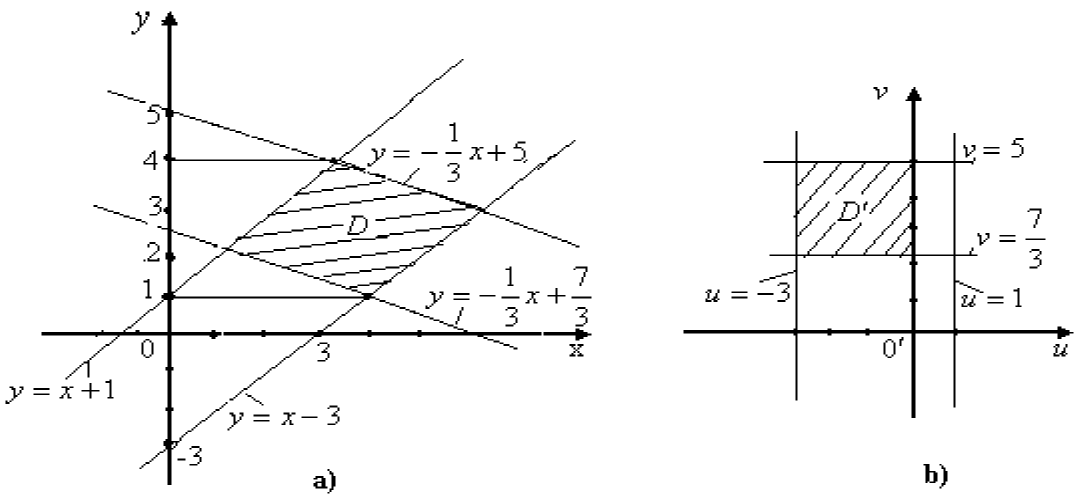
Demak,

$$|I'| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = -\frac{9}{16} - \frac{3}{16} = -\frac{12}{16} = -\frac{3}{4} \quad \text{va} \quad |I'| = \frac{3}{4}.$$

Shuning uchun:

$$\iint_D (y-x)dxdy = \iint_{D'} u \cdot \frac{3}{4}dudv = \frac{3}{4}\int_{-3}^{1}\left(\int_{\frac{7}{3}}^{5} udv\right)du = \frac{3}{4}\int_{-3}^{1} u\left(v\Big|_{\frac{7}{3}}^{5}\right)du = \frac{3}{4}\int_{-3}^{1} u\left(5 - \frac{7}{3}\right)du =$$

$$= \frac{3}{4}\cdot\frac{8}{3}\int_{-3}^{1} udu = 2\cdot\frac{u^2}{2}\Big|_{-3}^{1} = 1^2 - (-3)^2 = -8.$$

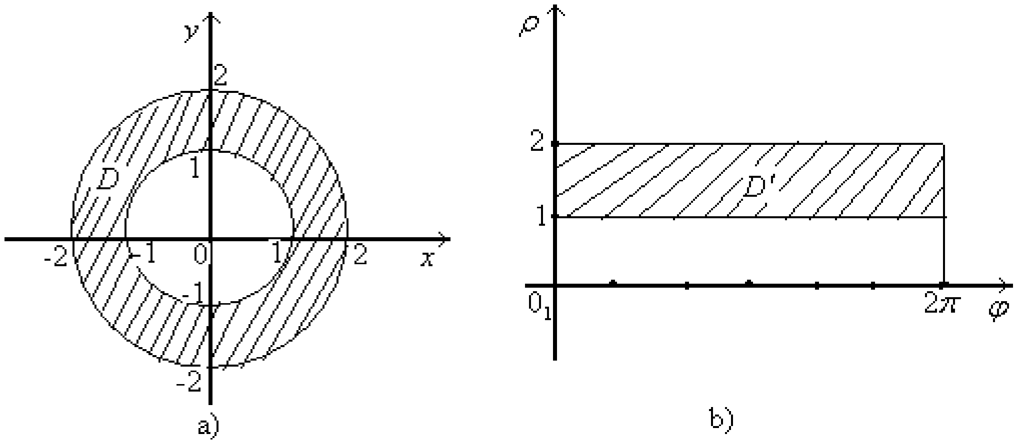


16-chizma.

2-misol. $I = \iint_D \frac{dxdy}{\sqrt{x^2 + y^2}}$ hisoblansin, bunda D soha $0xy$ tekislikning $x^2 + y^2 = 1$ va $x^2 + y^2 = 4$ konsentrik aylanalar bilan chegaralangan qismi (17-chizma).

Yechilishi. D soha noto'g'ri soha. Shuning uchun biz berilgan integralni x va y o'zgaruvchilar bo`yicha integrallamoqchi bo'lsak D sohani x=-1, x=1 to’g’ri chiziqlar yordamida to’rtta to’g’ri sohalarga ajratib hisoblashga to’g’ri kelar edi. Bu usul bilan berilgan integralni hisoblash ancha qiyin kechadi. Bu integralni o’zgaruvchilarni almashtirib, ya‘ni qutb koordinatalariga o’tib hisoblash ancha oson. $x=\rho\cos\theta,\ y=\rho\sin\theta$ almashtirishlar olinsa $\sqrt{x^2+y^2}=\rho$, $1\le\rho\le2$, $0\le\varphi\le2\pi$ bo’lib (1.4) formulaga ko’ra quyidagiga ega bo’lamiz

$$I=\iint\limits_{D^1}\frac{\rho d\theta d\rho}{\rho}=\iint\limits_{D^1}d\theta d\rho=\int_1^2\left(\int_0^{2\pi}d\theta\right)d\rho=\int_1^2\left(\theta\Big|_0^{2\pi}\right)d\rho=2\pi\int_1^2 d\rho=2\pi\rho\Big|_1^2=2\pi(2-1)=2\pi.$$

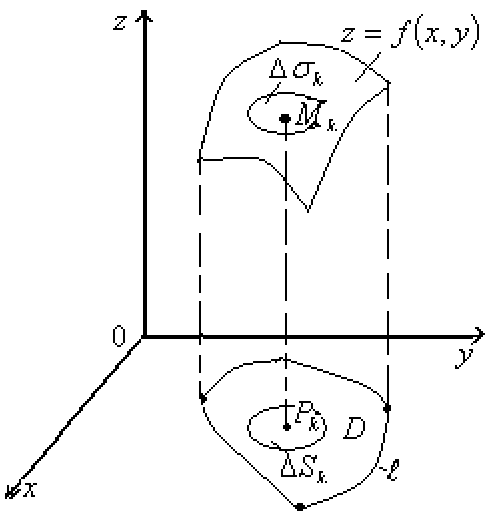


17-chizma.

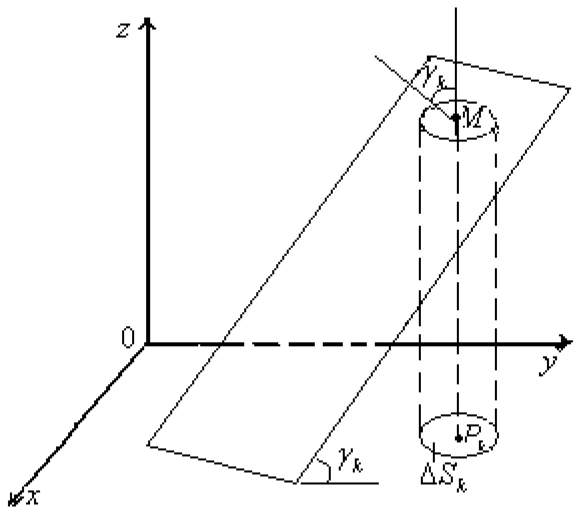
1.2. Sirt yuzini hisoblash

Aniq integralning tadbiqlarida aylanish jismining sirtini topish usuli bilan tanishgan edik. Endi $z=f(x,y)$ tenglama bilan berilgan istalgan silliq sirt bo‘lagining yuzini topish uchun formula chiqaramiz.

Sirtning Γ yopiq chiziq bilan chegaralangan qismining yuzini hisoblash talab etilsin (18-chizma); sirt o‘zining $z=f(x,y)$ tenglamasi bilan berilgan, bu yerdagi $f(x,y)$ qaralayotgan sohada uzluksiz va uzluksiz $f'_x,\ f'_y$ xususiy hosilalarga ega bo‘lgan funksiya.



18-chizma.



19-chizma.

Γ chiziqning $0xy$ tekislikdagi proyeksiyasini ℓ bilan belgilaymiz. $0xy$ tekislikdagi ℓ chiziq bilan chegaralangan sohani D bilan belgilaymiz. D sohani ixtiyoriy chiziqlar bilan n ta elementar $\Delta S_1, \Delta S_2, ..., \Delta S_n$ qismlarga bo‘lamiz. Har qaysi ΔS_k $(k = \overline{1,n})$ qismning ichida $P_k(x_k, y_k)$ nuqta olamiz. P_k nuqtaga sirt ustida $M_k(x_k, y_k, z_k = f(x_k, y_k))$ nuqta mos keladi. M_k nuqta orqali shu sirtga urinma tekislik o‘tkazamiz. Uning tenglamasi

$$z - z_k = f'_x(x_k, y_k)(x - x_k) + f'_y(x_k, y_k)(y - y_k) \qquad (1.5)$$

ko‘rinishga ega bo‘lishi ta’kidlangan edi. Urinma tekislikda, $0xy$ tekislikka ΔS_k yuz shaklida proyeksiyalanadigan $\Delta\sigma_k$ yuzni ajratamiz. Hamma $\Delta\sigma_k$ yuzlarning yig‘indisi

$$\sum_{k=1}^{n} \Delta\sigma_k$$

ni qaraymiz.

$\Delta\sigma_k$ yuzlarning diametrlaridan eng kattasi nolga intilganda bu yig‘indining σ limitini sirtning yuzi deb ataymiz, ya’ni ta’rifga binoan:

$$\sigma = \lim_{\max diam \Delta S_k \to 0} \sum_{k=1}^{n} \Delta\sigma_k . \qquad (1.6)$$

Endi sirtning yuzini hisoblashga kirishamiz. Urinma tekislik bilan $0xy$ tekislik orasidagi burchakni γ_k bilan belgilaymiz. U holda, urinma tekislikning $\Delta\sigma_k$ yuzining $0xy$ tekislikdagi proyeksiyasi ΔS_k yuzi orasida

$$\Delta S_k = \Delta\sigma_k \cos\gamma_k \qquad (1.7)$$

munosabat o‘rinli bo‘ladi. Bu formulaning uchburchak uchun to‘g‘riligi maktab geometriya kursida isbotlangan. Istalgan ko‘pburchakni bir nechta uchburchaklarga ajratish mumkin bo‘lganligi uchun bu formula ko‘pburchak uchun ham o‘rinli bo‘ladi. Istalgan egri chiziq bilan chegaralangan $\Delta\sigma_k$ tekis yuzni egri chiziqqa ichki chizilgan ko‘pburchaklarning limiti shaklida tasvirlanganligi uchun (1.7) formula istalgan tekis figura uchun o‘rinli bo‘ladi.

(1.7) formuladan

$$\Delta\sigma_k = \frac{\Delta S_k}{\cos\gamma_k}$$

ga ega bo‘lamiz, bunda $\gamma_k - \Delta\sigma_k$ va ΔS_k yuzlar orasidagi o‘tkir burchak. Bu yuzlar orasidagi γ_k burchak ularga perpendikulyarlar orasidagi burchakka, ya’ni sirtga M_k nuqtada o‘tkazilgan normal bilan $0z$ o‘q orasidagi burchakka teng.

Sirtga $M_k(x_k, y_k, z_k)$ nuqtasida o‘tkazilgan normal tenglamasi

$$\frac{x - x_k}{-f'_x(x_k, y_k)} = \frac{y - y_k}{-f'_y(x_k, y_k)} = \frac{z - z_k}{1}$$

ko‘rinishga ega bo‘lishi ta’kidlangan edi. Demak, $\vec{n}\{-f'_x(x_k, y_k); -f'_y(x_k, y_k); 1\}$ vektor normal to‘g‘ri chiziqning yo‘naltiruvchi vektori, ya’ni unga parallel vektor. Bu vektorning ya’ni normalning $0z$ o‘q bilan tashkil etgan burchagi γ_k ning kosinusi

$$\cos\gamma_k = \frac{n_x}{|\vec{n}|} = \frac{1}{\sqrt{1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)}}$$

bo‘ladi.

Demak, $\Delta\sigma_k = \sqrt{1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)}\Delta S_k$. Bu ifodani (1.6) tenglikka qo‘yib

$$\sigma = \lim_{\max diam\Delta S_k \to 0} \sum_{k=1}^{n} \sqrt{1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)}\Delta S_k$$

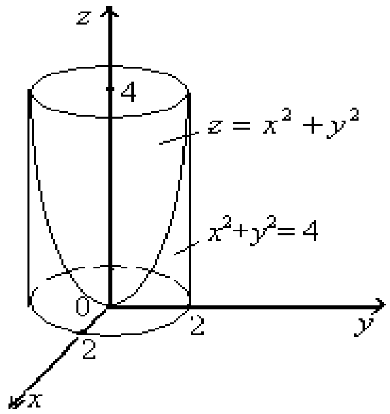
ni hosil qilamiz.

Bundan ikki o‘lchovli integralni ta’rifi hamda yopiq chegaralgan sohada uzluksiz funksiyaning integrallanuvchiligiga asoslanib, sirt yuzini hisoblash uchun

$$\sigma = \iint_D \sqrt{1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)}dxdy \qquad (1.8)$$

formulaga ega bo‘lamiz.

3-misol. $z = x^2 + y^2$ aylanish paraboloidining $x^2 + y^2 = 4$ silindr bilan kesilgan qismining sirti topilsin (20-chizma).



20-chizma

Yechilishi. (1.8) formuladan foydalanamiz. Bu yerda $f(x, y) = x^2 + y^2$; $f_x' = 2x, f_y' = 2y$ bo‘lgani uchun $\sqrt{1 + f_x'^2 + f_y'^2} = \sqrt{1 + 4(x^2 + y^2)}$.

Demak, $\sigma = \iint_D \sqrt{1 + 4(x^2 + y^2)}dxdy$, bu yerda D-markazi koordinatalar boshida bo‘lib radiusi 2 ga teng doira(240-chizma).

Integralni hisoblashni qutb koordinatalarida amalga oshiramiz:

$$\sigma = \iint_D \sqrt{1 + 4(x^2 + y^2)}dxdy = \iint_D \sqrt{1 + 4\rho^2}\rho d\rho d\theta = \int_0^{2\pi} d\theta \int_0^2 \sqrt{1 + 4\rho^2}\rho d\rho .$$

Ichki integral

$$\int_0^2 \sqrt{1 + 4\rho^2}\rho d\rho = \frac{1}{8}\int_0^2 (1 + 4\rho^2)^{\frac{1}{2}} d(1 + 4\rho^2) = \frac{1}{8} \cdot \frac{(1 + 4\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2 = \frac{1}{12} \cdot \left(17^{\frac{3}{2}} - 1\right) = \frac{1}{12} \cdot \left(17\sqrt{17} - 1\right).$$

Demak,

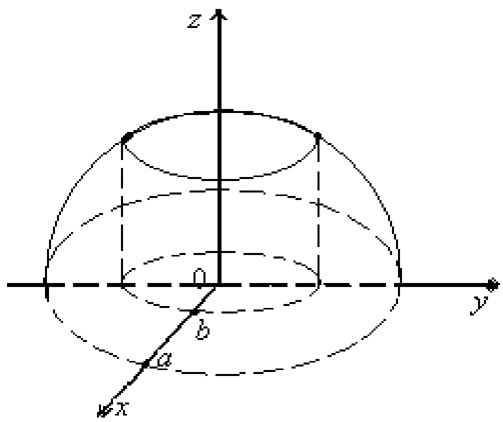
$$\sigma = \int_0^{2\pi} \frac{1}{12} \cdot \left(17\sqrt{17} - 1\right)d\theta = \frac{17\sqrt{17} - 1}{12} \cdot \theta\Bigg|_0^{2\pi} = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

4-misol. $x^2 + y^2 + z^2 = a^2$ sferaning $x^2 + y^2 = b^2$ $(b \le a)$ silindr bilan kesilgan qismining yuzi topilsin.

Yechilishi. Sferaning silindr ajratgan bo‘lagini birinchi aktantdagi qismining yuzi izlanayotgan σ yuzning $\frac{1}{8}$ qismini tashkil etadi. Shu bo‘lakning $0xy$ tekislikdagi proyeksiyasi $x^2+y^2=b^2$ aylana bilan chegaralangan doiraning to‘rtdan bir bo‘lagiga teng. Sfera tenglamasini z ga nisbatan yechsak

$$z=\sqrt{a^2-x^2-y^2}$$

bo‘ladi.



21-chizma.

$$z'_x=\frac{\left(a^2-x^2-y^2\right)'_x}{2\sqrt{a^2-x^2-y^2}}=-\frac{x}{z},\ z'_y=-\frac{y}{z},$$

$$\sqrt{1+z'^2_x+z'^2_y}=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=\sqrt{\frac{x^2+y^2+z^2}{z^2}}=\frac{a}{z}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

ga ega bo‘lamiz. (1.8) sirt yuzini topish formulasiga binoan

$$\frac{\sigma}{8}=\iint_D\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$$

bo‘ladi. Bunda D $x^2+y^2\le b^2$ doiraning birinchi chorakdagi qismi.

Integralni qutb koordinatalariga o‘tib hisoblaymiz

$$\frac{\sigma}{8}=\iint_{D'}\frac{a\rho d\theta d\rho}{\sqrt{a^2-\rho^2}}=\int_0^b\left(\int_0^{\frac{\pi}{2}}\frac{a\rho d\theta}{\sqrt{a^2-\rho^2}}\right)d\rho=a\int_0^b\frac{\rho d\rho}{\sqrt{a^2-\rho^2}}\cdot\int_0^{\frac{\pi}{2}}d\theta=\frac{a\pi}{2}\int_0^b\frac{\rho d\rho}{\sqrt{a^2-\rho^2}}=$$

$$=-\frac{a\pi}{2}\int_0^b\frac{d\left(a^2-\rho^2\right)}{2\sqrt{a^2-\rho^2}}=-\frac{a\pi}{2}\sqrt{a^2-\rho^2}\Big|_0^b=-\frac{a\pi}{2}\left(\sqrt{a^2-b^2}-\sqrt{a^2}\right)=\frac{a\pi}{2}\left(a-\sqrt{a^2-b^2}\right).$$

Demak, $\frac{\sigma}{8}=\frac{a\pi}{2}\left(a-\sqrt{a^2-b^2}\right).$

Bundan $\sigma=4a\pi\left(a-\sqrt{a^2-b^2}\right).$

Xususiy holda $a=b$ bo‘lganda oxirgi tenglikdan $\sigma=4a^2\pi$ sferaning yuzini topish formulasiga ega bo‘lamiz.

1.3. Ikki o‘lchovli integralning mexanikaga tadbiqlari

Moddiy bir jinsli bo‘lmagan D yupqa plastinkaning momentlari hamda og‘irlik markazini topishga kirishishdan oldin shu plastinkaning massasini topishda bajarilgan amallarni takrorlashni lozim topdik.

D plastinkaning sirt zichligi $\gamma=\gamma(x,y)$ plastinkaning $P(x,y)$ nuqtasining koordinatalarini uzluksiz funksiyasi sifatida berilganda D plastinkaning massasi m ni topish uchun D plastinkaning ixtiyoriy yo‘l bilan n ta kichik $\Delta S_1,\Delta S_2,...,\Delta S_n$ bo‘laklarga ajratib har bir ΔS_k $(k=\overline{1,n})$ bo‘lakda bittadan $P_k(x_k,\ y_k)$ nuqtani olamiz.

Agar ΔS_k bo‘lak juda kichik bo‘lsa, u holda zichlik P_k nuqtadagi $\gamma_k = \gamma_k(x_k, y_k)$ zichlikdan juda kam farq qiladi. D plastinkaning har bir ΔS_k bo‘lakda zichlik o‘zgarmas va u taqriban tanlangan P_k nuqtadagi zichlikka teng deb hisoblab ΔS_k bo‘lakning massasi Δm_k ni taqriban hisoblaymiz:

$$\Delta m_k \approx \gamma_k \Delta S_k = \gamma(x_k, y_k)\Delta S_k, \quad \left(k = \overline{1,n}\right)$$

bunda ΔS_k orqali ΔS_k bo‘lakning yuzi belgilangan. U holda D plastinkaning massasi

$$m = \sum_{k=1}^{n} \Delta m_k \approx \sum_{k=1}^{n} \gamma(x_k, y_k)\Delta S_k$$

taqribiy tenglik yordamida aniqlanar edi.

Izlanayotgan massaning aniq qiymati sifatida oxirgi yig‘indining $\max diam \Delta S_k \to 0$ dagi limiti qabul qilinadi. Bu limit D sohada uzluksiz $\gamma(x, y)$ funksiya uchun integral yig‘indi bo‘lganligi sababli u $\max diam \Delta S_k \to 0$ da chekli

$$m = \iint_D \gamma(x, y) dS$$

limitga ega bo‘ladi.

Bir so‘z bilan aytganda D plastinkaning $P(x, y)$ nuqtasini o‘z ichiga oluvchi ΔS elementar bo‘lagining massasi $\Delta m \approx \gamma(x, y)\Delta S$ ni butun D plastinka bo‘yicha integrallab plastinka massasini topish uchun

$$m = \iint_D \gamma(x, y) dS \qquad (1.9)$$

formulaga ega bo‘ldik.

Shunga o‘xshash Δm ni $0x$ va $0y$ koordinata o‘qlaridan y va x masofada joylashgan moddiy nuqta deb qarab

$$\Delta S_x \approx y\Delta m = y\gamma(x, y)\Delta S$$

va

$$\Delta S_y \approx x\Delta m = x\gamma(x, y)\Delta S$$

plastinkaning elementar statik momentlariga ega bo‘lamiz.

Bu ifodalarni butun D plastinka bo`yicha integrallab D plastinkaning koordinata o‘qlariga nisbatan statik momentlarini topamiz:

$$\left.\begin{aligned} S_x &= \iint_D y\gamma(x, y) dS, \\ S_y &= \iint_D x\gamma(x, y) dS \end{aligned}\right\} \qquad (1.10)$$

Agar D plastinkaning og‘irlik markazi koordinatalarini (x_C, y_C) orqali belgilasak

$$S_y = m \cdot x_C, \; S_x = m \cdot y_C$$

yoki bundan

$$\left.\begin{aligned} x_C &= \frac{S_y}{m} = \frac{1}{m}\iint_D x\gamma(x, y) dS, \\ y_C &= \frac{S_x}{m} = \frac{1}{m}\iint_D y\gamma(x, y) dS \end{aligned}\right\} \qquad (1.11)$$

bo‘ladi, bunda m-massa (1.9) formula yordamida aniqlanadi.

Shunga o‘xshash D plastinkaning $0x$ va $0y$ koordinata o‘qlariga nisbatan elementar inersiya momentlari uchun

$$\Delta I_x = y^2 \Delta m \approx y^2 \gamma(x, y)\Delta S ,$$
$$\Delta I_y = x^2 \Delta m \approx x^2 \gamma(x, y)\Delta S$$

ifodalarga ega bo‘lamiz.

Bularni butun D plastinka bo‘yicha integrallab plastinkaning koordinata o‘qlariga nisbatan **inersiya momentlarini** topish formulalari

$$\left.\begin{aligned} I_x &= \iint_D y^2 \gamma(x, y) dS, \\ I_y &= \iint_D x^2 \gamma(x, y) dS \end{aligned}\right\} \qquad (1.12)$$

ni hosil qilamiz.

Koordinatalar boshiga nisbatan **elementar inersiya momenti**

$$\Delta I_0 = r^2 \Delta m \approx \left(x^2 + y^2\right)\gamma(x, y)\Delta S$$

formula yordamida topiladi, bunda $r^2 = x^2 + y^2 - \Delta m$ massadan koordinatalar boshigacha masofaning kvadrati. Oxirgi ifodani D plastinka bo‘yicha integrallab plastinkaning koordinata boshiga nisbatan **inersiya momentini** topish formulasi

$$I_0 = \iint_D \left(x^2 + y^2\right)\gamma(x, y) dS \qquad (1.13)$$

ga ega bo‘lamiz. (1.12) va (1.13) formulalardan

$$I_0 = I_x + I_y \qquad (1.14)$$

ekani kelib chiqadi.

Chiqarilgan formulalarda $\gamma(x, y) \equiv 1$ deb faraz qilinsa D geometrik figura uchun yaroqli formulalar hosil bo‘ladi.

Shuni ta’kidlash joizki hisoblash to‘g‘ri burchakli dekart koordinatalarida amalga oshirilganda $dS = dxdy$ deb va hisoblash qutb koordinatalarida amalga oshirilganda $dS = \rho d\theta d\rho$ deb olinadi, bunda θ-qutb burchagi, ρ- qutb radiusi.

5-misol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning birinchi chorakda yotgan qismi hamda koordinata o‘qlari bilan chegaralangan figuraning $0x$ va $0y$ o‘qlarga nisbatan statik momentlari S_x, S_y topilsin.

Yechilishi. Ellips tenglamasini y ga nisbatan yechib birinchi chorakdagi ellips bo‘lagi uchun $y = \frac{b}{a}\sqrt{a^2 - x^2}$ tenglamaga ega bo‘lamiz. Integrallash sohasi D da y o‘zgaruvchi 0 dan $\frac{b}{a}\sqrt{a^2 - x^2}$ gacha; x o‘zgaruvchi 0 dan a gacha o‘zgaradi. Shuning uchun (1.10) formulalardan $\gamma(x, y) \equiv 1$ bo‘lganda

$$S_x = \iint_D y dx dy = \int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} y dy = \int_0^a \left(\frac{y^2}{2} \Bigg|_0^{\frac{b}{a}\sqrt{a^2 - x^2}} \right) dx = \int_0^a \left(\frac{1}{2} \frac{b^2}{a^2} \left(a^2 - x^2\right) \right) dx =$$

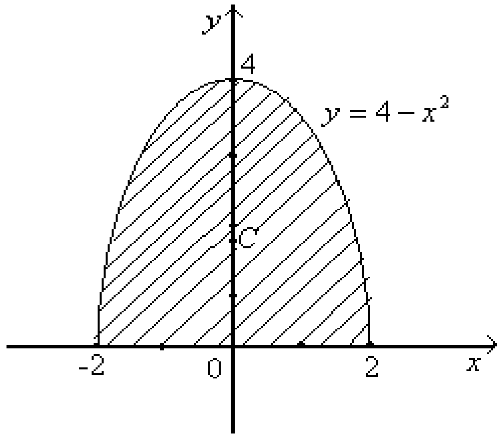
$$= \frac{b^2}{2a^2}\int_0^a \left(a^2 - x^2\right)dx = \frac{b^2}{2a^2}\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{b^2}{2a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{b^2a}{3};$$

$$S_y = \iint_D xdxdy = \int_0^a xdx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,xdx = -\frac{b}{2a}\int_0^a \sqrt{a^2 - x^2}\,d\left(a^2 - x^2\right) =$$

$$= -\frac{b}{2a}\,\frac{\left(a^2 - x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^a = \frac{b}{3a}\left(a^2\right)^{\frac{3}{2}} = \frac{a^2b}{3}$$

ga ega bo‘lamiz. Demak, $S_x = \frac{b^2a}{3}$, $S_y = \frac{a^2b}{3}$.

6-misol. $y = 4 - x^2$ parabola va $0x$ o‘q bilan chegaralangan bir jinsli figura yuzining og‘irlik markazini koordinatalari topilsin.



22-chizma.

Yechilishi. Figura $0y$ o‘qqa nisbatan simmetrik bo‘lgani uchun uning og‘irlik markazi $0y$ o‘qda yotadi, ya’ni $x_C = 0$. Qaralayotgan holda $\gamma(x, y) \equiv 1$ ekanini hisobga olib og‘irlik markazining y_C ordinatasini (1.11) formulaning ikkinchisiga asoslanib topamiz. Buning uchun S_x statik momentni topamiz:

$$S_x = \iint_D ydxdy = \int_{-2}^{2} dx \int_0^{4-x^2} ydy = \int_{-2}^{2}\left(\frac{y^2}{2}\Bigg|_0^{4-x^2}\right)dx = \frac{1}{2}\int_{-2}^{2}\left(4 - x^2\right)^2 dx = \frac{1}{2}\int_{-2}^{2}\left(16 - 8x^2 + x^4\right)dx =$$

$$= \frac{1}{2}\left(16x - 8\cdot\frac{x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-2}^{2} = \frac{1}{2}\left(16\cdot 4 - \frac{8}{3}\left(2^3 + 2^2\right) + \frac{2^5 + 2^5}{5}\right) = \frac{256}{15},$$

$$m = \iint_D dxdy = \int_{-2}^{2} dx \int_0^{4-x^2} dy = \int_{-2}^{2}\left(4 - x^2\right)dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = 4\cdot 2 - \frac{2^3}{3} + 4\cdot 2 - \frac{2^3}{3} = 16 - \frac{16}{3} = \frac{32}{3}.$$

(1.11) formulaning ikkinchisiga asosan

$$y_C = \frac{\iint_D ydxdy}{\iint_D dxdy} = \frac{\frac{256}{15}}{\frac{32}{3}} = \frac{8}{5}.$$

Demak, $C\left(0; \frac{8}{5}\right)$ qaralayotgan figuraning og‘irlik markazidir.

7-misol. Zichlik $\gamma(x, y) \equiv 1$ deb hisoblab 22-chizmada tasvirlangan figura yuzining $0y$ o‘qqa nisbatan inersiya momenti I_y topilsin.

Yechilishi. (1.12) formulaning ikkinchisiga binoan

$$I_y = \iint_D x^2dxdy = \int_{-2}^{2} dx \int_0^{4-x^2} x^2dy = \int_{-2}^{2} x^2dx \int_0^{4-x^2} dy = \int_{-2}^{2} x^2\left(4 - x^2\right)dx = \int_{-2}^{2}\left(4x^2 - x^4\right)dx =$$

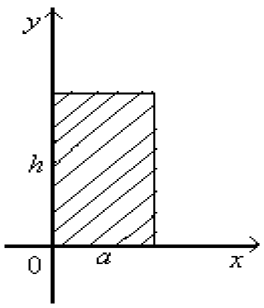
$$= \left(4 \cdot \frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_{-2}^{2} = \frac{4}{3} \cdot 2 \cdot 2^3 - \frac{2 \cdot 2^5}{5} = \frac{128}{15}.$$

8-misol. Asosi a sm. balandligi h sm bo‘lgan to‘g‘ri to‘rtburchakning asosi va balandligiga nisbatan inersiya momentlari topilsin.

Yechilishi. To‘g‘ri to‘rtburchakli koordinatalar sistemasini 23-chizmada ko‘rsatilgandek tanlaymiz. U holda to‘g‘ri to‘rtburchakning asosi va balandligiga nisbatan intersiya momentlari uning $0x$ va $0y$ o‘qlarga nisbatan inersiya momentlari bo‘ladi. $\gamma(x, y) \equiv 1$ deb hisoblab (1.12) formulaga binoan quyidagilarni hosil qilamiz:

$$I_x = \iint_D y^2 dxdy = \int_0^a dx \int_0^h y^2 dy = \frac{h^3}{3} \int_0^a dx = \frac{ah^3}{3} sm^4;$$

$$I_y = \iint_D x^2 dxdy = \int_0^a x^2 dx \int_0^h dy = h \int_0^a x^2 dx = \frac{a^3 h}{3} sm^4.$$



23-chizma.

9-misol. 23-chizmada tasvirlangan to‘g‘ri to‘rtburchakning 0 uchiga nisbatan inersiya momenti I_0 topilsin.

Yechilishi. (1.14) formulaga binoan

$$I_0 = I_x + I_y = \frac{ah^3}{3} sm^4 + \frac{a^3 h}{3} sm^4 = \frac{ah}{3}\left(h^2 + a^2\right) sm^4.$$