1-Amaliy mavzuga: Kompleks son. Uning geometrik tasviri, moduli va argumenti. Kompleks sonlar ustida amallar.

Haqiqiy a sonini mavhum bi soniga qo‘shishdan $a+bi$ kompleks sonni hosil qilamiz. Demak, $a+bi$ ifpdaga ***kompleks son*** deyiladi (bundaa, b haqiqiy sonlar, i esa mavhum birlik, a-kompleks sonning haqiqiy qismi, bi- mavhum qismlari). Agar $a_1+b_1 i$ va $a_2+b_2 i$ kompleks sonlarda
$a_1=a_2;\ b_1=b_2$ bo‘lsa, ular teng deyiladi.

$$\boldsymbol{z=a+bi.}$$

$z=a+bi$ va $z=a-bi$ komleks sonlar ***qo‘shma*** kompleks sonlar deyiladi. Haqiqiy va mavhum qismlarning ishoralari bilan farq qiluvchi ikkita $z_1=a+bi$ va $z_2=-a-bi$ kompleks sonlar ***qarama-qarshi*** komleks sonlar deyiladi.

$\boldsymbol{z=x+yi}$ ko‘rinishdagi son algebraik ko‘rinishdagi kompleks son deyiladi. $x=rcos\varphi$; $y=rsin\varphi$; bunda $r-$kompleks soni z ni tasvirlagan vektorning uzunligini ifodalaydi va unga z sonning moduli, φ burchak esa z ning ***argumenti*** deyiladi.

$$|z|=|x+yi|=r=\sqrt{x^2+y^2}$$

$$z=x+yi => \ r(cos\varphi+isin\varphi)$$

$$r=\sqrt{x^2+y^2}$$

$1-misol$: Kompleks sonning moduli 3ga, argumenti $\varphi=\frac{\pi}{4}$ ga teng bo‘lsa, uning haqiqiy va mavhum qismlarini toping.

Yechilishi: $x=rcos\varphi\ \ va\ \ y=rsin\varphi$ (1) formuladan foydalansak,

$$x=rcos\varphi=3cos\frac{\pi}{4}=3\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2},$$

$$y=rsin\varphi=3sin\frac{\pi}{4}=3\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}.$$

$2-misol$: $z=i$ kompleks sonning argumentini toping.

Yechilishi: $x=0; y=1; r=1;\ \varphi=\frac{\pi}{2}$.

$$y=rsin\varphi=3sin\frac{\pi}{4}=3\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}.$$

$3-misol$: $z_1=2+5i$ va $z_2=-1-3i$ kompleks sonlarning yig‘indisini toping.

Yechilishi:

$$z_1+z_2=(2+5i)+(-1-3i)=(2-1)+i(5-3)=1+2i.$$

$4-misol$: $z_1=6+5i$ va $z_2=4-2i$ kompleks sonlarning ayirmasini toping.

Yechilishi:

$$z_1 - z_2 = (6+5i) - (4-2i) = (6-4) + i(5+2) = 2+7i.$$

$5-misol$:

$z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$ va $z_2 = \sqrt{2}\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right)$ kompleks konlarning ko'paytmasini toping.

Yechilishi:

$$z_1 \cdot z_2 = 2\sqrt{2}\left[cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + i\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right] = 2\sqrt{2}\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right) =$$
$$= 2\sqrt{2}i.$$

$6-misol$: $z_1 = \sqrt{3} + i$ ni $z_2 = -3-3i$ kompleks songa bo'ling.

Yechilishi: $\frac{z_1}{z_2} = \frac{\sqrt{3}+i}{-3-3i} = \frac{(\sqrt{3}+i)\cdot(-3+3i)}{(-3-3i)\cdot(-3+3i)} = \frac{-3\sqrt{3}-3+(\sqrt{3}-3)i}{9+9} =$

$$= \frac{-3\left[\sqrt{3}+1-(\sqrt{3}-1)i\right]}{18} = \frac{-\sqrt{3}-1}{6} + \frac{\sqrt{3}-1}{6}i.$$

$7-misol$: $z_1 = \sqrt{3} + i = 2\left(cos\frac{\pi}{6} + i\,sin\frac{\pi}{6}\right)$ ni

$z_2 = -3-3i = 3\sqrt{2}\left(cos\frac{5\pi}{4} + i\,sin\frac{5\pi}{4}\right)$ kompleks songa bo'ling.

Yechilishi:

$$\frac{z_1}{z_2} = \frac{2\left(cos\frac{\pi}{6}+i\,sin\frac{\pi}{6}\right)}{3\sqrt{2}\left(cos\frac{5\pi}{4}+i\,sin\frac{5\pi}{4}\right)} = \frac{2}{3\sqrt{2}}\left[cos\left(\frac{\pi}{6}-\frac{5\pi}{4}\right) + i\,sin\left(\frac{\pi}{6}-\frac{5\pi}{4}\right)\right] =$$
$$= \frac{2}{3\sqrt{2}}\left[cos\left(-\frac{13\pi}{12}\right) + i\,sin\left(-\frac{13\pi}{12}\right)\right] = \frac{2}{3\sqrt{2}}\left(cos\frac{13\pi}{12} - i\,sin\frac{13\pi}{12}\right) =$$
$$= \frac{2}{3\sqrt{2}}\left[cos\left(\pi+\frac{\pi}{12}\right) + i\,sin\left(\pi+\frac{\pi}{12}\right)\right] = \frac{2}{3\sqrt{2}}\left(-cos\frac{\pi}{12} + i\,sin\frac{\pi}{12}\right) =$$
$$== \frac{2}{3\sqrt{2}}\left[-cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + i\,sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right] =$$
$$= \frac{2}{3\sqrt{2}}\left[\left(-cos\frac{\pi}{3}\cdot cos\frac{\pi}{4} - sin\frac{\pi}{3}\cdot sin\frac{\pi}{4}\right) + i\left(sin\frac{\pi}{3}\cdot cos\frac{\pi}{4} - cos\frac{\pi}{3}\cdot sin\frac{\pi}{4}\right)\right] =$$
$$= \frac{2}{3\sqrt{2}}\left[\left(-\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)\right] = \frac{\sqrt{3}+1}{6} + i\frac{\sqrt{3}-1}{6}.$$

Kompleks sonni darajaga ko'tarish:

$z = r(cos\varphi + isin\varphi)$ kompleks son uchun n natural bo'lganda

$z^n = r^n(cosn\varphi + isinn\varphi)$ (Muavr formulasi).

$$\boldsymbol{i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = -1, \quad i^{4k+3} = -i.}$$

$8-misol$: $\sqrt[5]{\mathbf{1}}$ ning ildizlarini toping.

Yechilishi: $\sqrt[5]{1}$ sonni trigonometrik ko'rinishda yozamiz.

$z = 1$ bo'lib, $z = 1 = cos0 + i\sin 0$ bo'ladi.

$$\sqrt[5]{1} = \sqrt[5]{cos0 + i\,sin\,0} = cos\frac{2\pi k}{5} + i\,sin\frac{2\pi k}{5}.$$

$$k_0 = 0;\ z_1 = cos0 + i\sin 0 = 1;$$

$$k_1 = 1;\ z_2 = cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = cos72° + i\,sin72° \approx 0{,}309 + i0{,}951;$$

$$k_2 = 2;\ z_3 = cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = cos144° + i\,sin144° \approx 0{,}809 + i0{,}587;$$

$$k_3 = 2;\ z_4 = cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = cos216° + i\,sin216° \approx 0{,}809 - i0{,}587;$$

$9-misol$: $(-1+i)^5$ ni hisoblang.

Yechilishi: $z = -1 + i = \sqrt{2}\left(cos\frac{3\pi}{4} + i\,sin\frac{3\pi}{4}\right);$

$$z^5 = (-1+i)^5 = \sqrt{2}^5 \cdot \left(cos\frac{3\pi}{4} + i\,sin\frac{3\pi}{4}\right)^5 =$$

$$= 4\sqrt{2}\left(cos5\cdot\frac{3\pi}{4} + i\,sin5\cdot\frac{3\pi}{4}\right) = 4\sqrt{2}(cos675^{\circ} + isin675°) =$$

$$= 4\sqrt{2}[cos(720° - 45°) + i\,sin(720° - 45°)] = 4\sqrt{2}(cos45° - i\,sin45°) =$$

$$= 4\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 4 - 4i = 4(1 - i).$$

$10-misol$: $(1-i)^{10}$ ni hisoblang.

Yechilishi: $(1-i)^{10} = \left[\sqrt{2}\left(cos\frac{7\pi}{4} + i\,sin\frac{7\pi}{4}\right)\right]^{10} =$

$$= 2^5\left(cos\frac{5\cdot 7}{4} + i\,sin\frac{5\cdot 7}{4}\right) = 2^5\left(cos\frac{35}{4} + i\,sin\frac{35}{4}\right) =$$

$$= 32\left(cos\left(16 + \frac{3\pi}{2}\right) + i\,sin\left(16\pi + \frac{3\pi}{2}\right)\right) = 32\left(cos\frac{3\pi}{2} + i\,sin\frac{3\pi}{2}\right) =$$

$$= 32\cdot(-i) = -32i$$

Javob: $(1-i)^{10} = -32i$

$11-misol$: $z_1 = 1 + 2i\ va\ z_2 = 1 - i$ kompleks sonlarning ayirmasi va bo‘linmasini toping.

Yechilishi: $z_1 - z_2 = (1+2i) - (1-i) = (1-1) + i(2+1) = 3i.$

$$\frac{z_1}{z_2} = \frac{(1+2i)\cdot(1+i)}{(1-i)\cdot(1+i)} = \frac{1 + 2i + i - 2}{1 - i + i + 1} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i.$$

$12-misol$: $z = -35 - 12i$ kompleks sonning kvadrat ildizlarini toping.

Yechilishi: Bu yerda $a = -35, b = -12$ ekanligidan

$$\sqrt{a^2+b^2}=\sqrt{(-35)^2+(-12)^2}=\sqrt{1225+144}=\sqrt{1369}=37.$$

$$u^2=\frac{1}{2}(35+37)=36;\quad v^2=\frac{1}{2}(-35+37)=1.$$

$u=\pm 6;\quad v=\pm 1$ hamda $b<0$ bo‘lganligi sababli, u va v larning ishorasiturli xil bo‘ladi, shuning uchun

$$\sqrt{-35-12i}=\pm(6-i).$$

13$-misol$: $\alpha=1-i$ kompleks sonni trigonometrik shaklga keltiring.

Yechilishi: Bunda $a=1, b=-1$ ekanligidan $r=\sqrt{1+1}=\sqrt{2}$. U holda $cos\varphi=\frac{1}{\sqrt{2}},\ sin\varphi=-\frac{1}{\sqrt{2}}$; tenglikdan $\varphi=\frac{7\pi}{4}$ ga ega bo‘lamiz.

Natijada $\alpha=\sqrt{2}\left(cos\frac{7\pi}{4}+i\ sin\frac{7\pi}{4}\right)$.

Javob: $\alpha=\sqrt{2}\left(cos\frac{7\pi}{4}+i\ sin\frac{7\pi}{4}\right)$

14$-misol$: $\alpha=1-i\ va\ \beta=\sqrt{2}\left(cos\frac{\pi}{8}+i\ sin\frac{\pi}{8}\right)$ kompleks sonlarning ko‘paytmasini toping.

Yechilishi: Bunda $\alpha=\sqrt{2}\left(cos\frac{7\pi}{4}+i\ sin\frac{7\pi}{4}\right)$ ekanligini hisobga olsak.

$$\alpha\cdot\beta=\sqrt{2}\left(cos\frac{7\pi}{4}+i\ sin\frac{7\pi}{4}\right)\cdot\sqrt{2}\left(cos\frac{\pi}{8}+i\ sin\frac{\pi}{8}\right)=$$

$$=2\left(cos\frac{15\pi}{8}+i\sin\frac{15\pi}{8}\right).$$

Javob: $\alpha\cdot\beta=2\left(cos\frac{15\pi}{8}+i\sin\frac{15\pi}{8}\right)$.

15$-misol$: $\alpha=1-i\ va\ \beta=\sqrt{2}\left(cos\frac{\pi}{8}+i\ sin\frac{\pi}{8}\right)$ kompleks sonlarning ko‘paytmasini toping.

Yechilishi: Bunda

2-Amaliy mashg‘ulot: **Ko‘p o‘zgaruvchili funksiya, uning aniqlanish sohasi, limit va uzluksizligi. Xususiy hosilalar. To‘la differensial.**

$z = z(x,y),\ \ z = \varphi(x,y),\ z = F(x,y)\ yoki\ z = \ f(x,y)$ $\quad x = \ x_0,\ y = y_0$ larga mos z_0 xususiy qiymat $z_0 = z_{\left|\substack{x=x_0 \\ y=y_0}\right|}$ yoki $z_0 = f(x_0, y_0)$.

1-misol: $x = -1, y = 2\ \ da\ z = x^3 + y^3$ funksiyaning qiymatini toping.

$Yechish:\quad z_{\left|\substack{x=x_0 \\ y=y_0}\right|} = (-1)^3 + 2^3 = -1 + 4 = 3.$

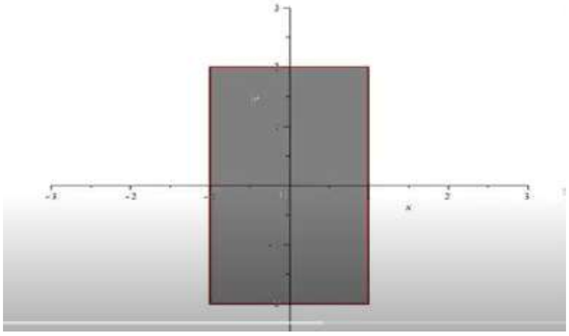
2-misol: $z = \sqrt{4 - x^2 + y^2}$ funksiya uchun E=?

Yechish: $4 - x^2 + y^2 \geq 0$ bundan $4 \leq x^2 + y^2 \geq 0 \Rightarrow\ x^2 + y^2 = 4\ u\ holda\ E = [0;2)$.

Ikki o‘zgaruvchili funksiyaning aniqlanish sohasini toping.

3-misol: $= \sqrt{1 - x^2} + \sqrt{4 - y^2}$.

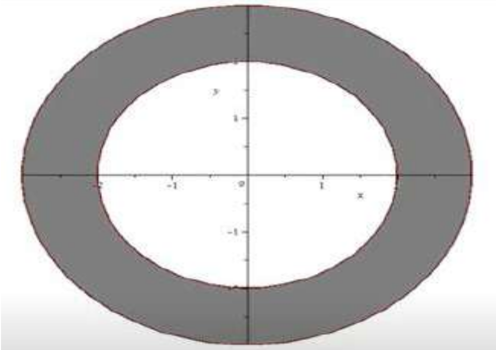
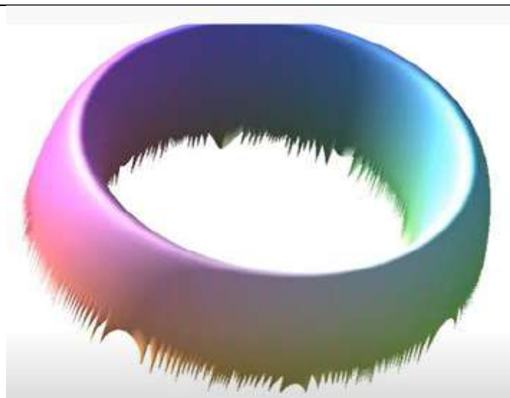
$Yechish:\ 1 - x^2 \geq 0, 4 - y^2 \geq 0\ \rightarrow x \in [-1; 1], y \in [-2; 2]$

4-misol: $u = \sqrt{(x^2 + y^2 - 4)(9 - x^2\ - y^2)}$.

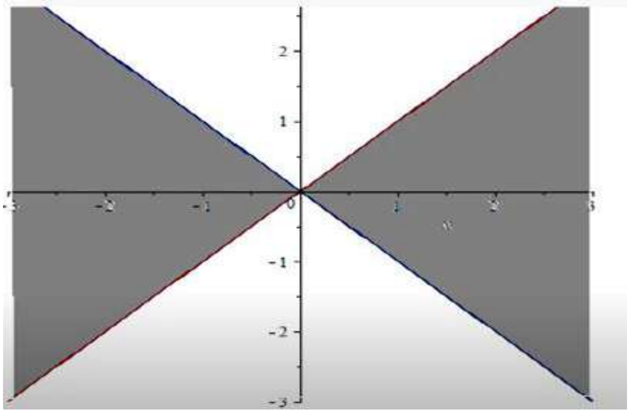
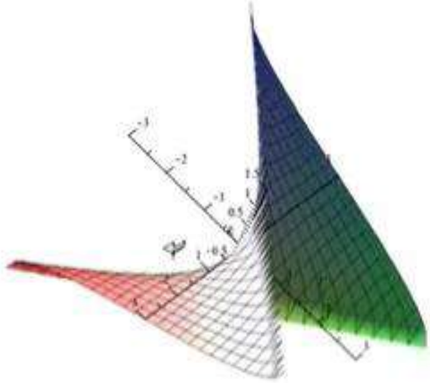
$Yechish: (x^2 + y^2 - 4)(9 - x^2\ - y^2) \rightarrow [x^2 + y^2 = t] \rightarrow (t - 4)(9 - t) \geq 0\ \rightarrow 4 \leq t \leq 9\ \rightarrow$

$2^2 \leq x^2 + y^2 \leq 3^2.$

5-misol: $u = arcsin\frac{y}{x}$.

$Yechish:\ -1 \leq \frac{y}{x} \leq 1\ \ \rightarrow x \geq 0 \rightarrow -x \leq y \leq x; x < 0\ \rightarrow x \leq y \leq -x.$

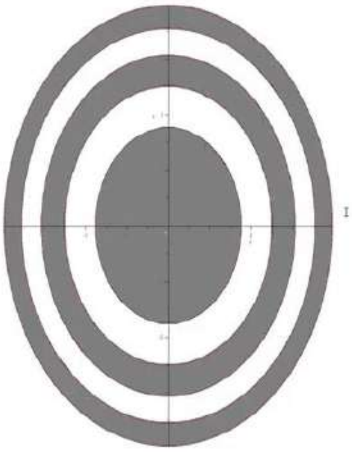
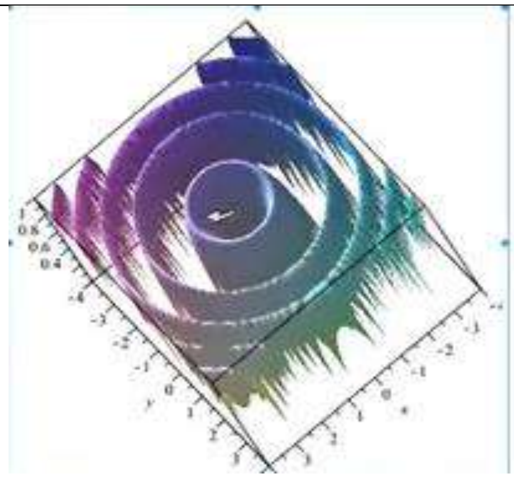
6-misol: $u = \sqrt{sin(x^2+y^2)}.$

$Yechish\!:\ sin(x^2+y^2) \geq 0 \quad \rightarrow \quad 2\pi n \leq x^2+y^2 \leq \pi + 2\pi n$

$n = 0 \quad \rightarrow \quad 0 \leq x^2+y^2 \leq \pi.$

$n = 1 \quad \rightarrow \quad 2\pi \ \leq x^2+y^2 \leq 3\pi.$

$n = 2 \quad \rightarrow \quad 4\pi \leq x^2+y^2 \leq 5\pi.$

7-misol: $A = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ limitni hisoblang.

Yechish: Limit belgisi ostidagi ifodani elementar almashtirishlar yordamida soddalshtirib topamiz.

$$A = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{\left(\sqrt{x^2+y^2+1}-1\right)\left(\sqrt{x^2+y^2+1}+1\right)} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2+1-1}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \sqrt{x^2+y^2+1}+1 = 2$$

8-misol: $z = arcctg\,\frac{x}{y}$ funksiyaning xususiy hosilalarini toping.

Yechish: Xususiy hosilalarni topish formulasidan foydalanamiz.

$$\lim_{\Delta x\to 0} \frac{\Delta_x z}{\Delta x} \equiv \frac{\partial z}{\partial x} = z_x^{`} = f_x^{`}(x,y) \qquad \lim_{\Delta y\to 0} \frac{\Delta_x z}{\Delta y} \equiv \frac{\partial z}{\partial y} = z_y^{`} = f_y^{`}(x,y) \quad (1)$$

$$\frac{\partial z}{\partial x} = -\frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{y}{x^2}\right) = \frac{y}{x^2+y^2} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = -\frac{x}{x^2+y^2}$$

9-misol: $u = ln^2(x^2+y^2+z^2)$ funksiyaning xususiy hosilalarini toping.

Yechish: $\frac{\partial z}{\partial x} = 2ln(x^2+y^2+z^2) \cdot \frac{1}{x^2+y^2+z^2} \cdot 2x = \frac{4xln(x^2+y^2+z^2)}{x^2+y^2+z^2}$,

$\frac{\partial z}{\partial y} = 2ln(x^2+y^2+z^2) \cdot \frac{1}{x^2+y^2+z^2} \cdot 2y = \frac{4yln(x^2+y^2+z^2)}{x^2+y^2+z^2}$,

$\frac{\partial z}{\partial z} = 2ln(x^2+y^2+z^2) \cdot \frac{1}{x^2+y^2+z^2} \cdot 2z = \frac{4zln(x^2+y^2+z^2)}{x^2+y^2+z^2}$.

10-misol: $u = (xy^2)^{z^3}$ xususiy differinsiallarini toping.

Yechish: $d_x z = f_x^{`}(x,y)dx,\ d_y z = f_y^{`}(x,y)dy$ xususiy differinsiallarni topish formulasidan foydalanamiz.

$d_x u = z^3 \cdot (xy^2)^{z^3-1} \cdot y^2 dx,$

$d_y u = z^3 \cdot (xy^2)^{z^3-1} \cdot 2xydy,$

$d_z u = (xy^2)^{z^3} \cdot \ln(xy^2) \cdot 3z^2 dz.$

11-misol: $u = ln^2(x^2+y^2-z^2)$ funksiyaning to‘la differinsiallini toping.

Yechish: Dastlab funksiyaning xususiy hosilalarini topamiz:

$\frac{\partial z}{\partial x} = 2ln(x^2+y^2-z^2) \cdot \frac{1}{x^2+y^2-z^2} \cdot 2x = \frac{4xln(x^2+y^2-z^2)}{x^2+y^2-z^2}$,

$\frac{\partial z}{\partial y} = 2ln(x^2+y^2-z^2) \cdot \frac{1}{x^2+y^2-z^2} \cdot 2y = \frac{4yln(x^2+y^2-z^2)}{x^2+y^2-z^2}$,

$\frac{\partial z}{\partial z} = 2ln(x^2+y^2-z^2) \cdot \frac{1}{x^2+y^2-z^2} \cdot (-2z) = -\frac{4zln(x^2+y^2-z^2)}{x^2+y^2-z^2}$.

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y}$dy+$\frac{\partial u}{\partial z} dz$ to‘la differinsiallini topish formulasi bo‘yicha hisoblasak:

$du = \frac{4xln(x^2+y^2-z^2)}{x^2+y^2-z^2} \cdot (xdx + ydx + zdz)$ ga ega bo‘lamiz.

12-misol: $u = \sqrt{x^2+y^2+z^2}$ funksiyaning x, y, z o‘zgaruvchilarga nisbatan xususiy hosilalarining P(2;-2;1) nuqtadagi qiymatini hisoblang.

Yechish: Xususiy hosilalarni topamiz: $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} - yz,$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} - xz, \qquad \frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} - xy.$$

Bu ifodalarga berilgan nuqtaning koordinatalarini qo‘yamiz:

$\left.\frac{\partial u}{\partial x}\right|_P = \frac{2}{3} + 2 = \frac{8}{3}$, $\quad \left.\frac{\partial u}{\partial y}\right|_P = -\frac{2}{3} - 2 = -\frac{8}{3}$, $\quad \left.\frac{\partial u}{\partial z}\right|_P = \frac{1}{3} + 4 = \frac{13}{3}$.

13-misol: Agar $u = x+y^2+z^3$ funksiyada $y = sinx, z = cosx$ bo‘lsa, uning to‘la hosilasini toping.

Yechish: To‘la hosilani topish formulasidan foydalansak:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx} = 1 + 2ycosx + 3z^2(-sinx) =$$

$$= 1 + 2sinxcosx - 3cos^2x\ sinx.$$

Mustaqil yechish uchun misollar:

Quyidagi funksiyalarning aniqlanish sohasini toping.

1) $z = \sqrt{y^2 - 2x + 4}$.
2) $z = \sqrt{4 - x^2 + y}$
3) $z = \sqrt{x^2 - y^2 - 9}$.
4) $z = \sqrt{4 - y^2 + x}$.

Quyidagi funksiyalarning xususiy hosilalarini toping.

5) $z = (x^3+y^3-xy^2)$.
6) $z = (x^3+y^3-xy^2)^3$.
7) $z = arcsin\frac{x}{y}$.
8) $u = arctg\frac{x-y}{z}$.

Quyidagi funksiyalarning xususiy differensiallarini toping.

9) $u = x^{yz}$.
10) $u = \frac{x^2+y^2+z^2}{z^2-x^2-y^2}$.
11) $z = \ln\sqrt{x^2 + y^2}$.
12) $u = tg^2(x - y^2 + z)$.

Quyidagi funksiyalarning to‘la differensiallarini toping.

13) $z = x^3+xy^2+x^2y$.
14) $z = e^{x^2-y^2}$.
15) $z = ln^2(x^2+y^2)$.

Quyidagi aniq integrallarni hisoblang.

16) $\int_2^3 x\,dx =$
17) $\int_0^2 x^3\ dx =$
18) $\int_{-1}^3 (x^2 - 4 + 5)\,dx =$
19) $\int_1^2 (3x - 2)\,dx =$
20) $\int_{\frac{\pi}{2}}^{\pi} sinx\,dx =$
21) $\int_0^{\pi} cosx\,dx =$
22) $\int_0^2 x^6 =$

5-Amaliy mavzuga: Aniqmas integrallarga doir misollar yechish.

1.Quyidagi integrallarni hisoblang: $a)\ \int 3^{5x}dx \quad b)\ \int 4^{2x}dx \quad v)\ \int 5^{2x}\,dx$ $g)\ \int 19^{2x}dx \qquad d)\ \int 7^{2x}\,dx.$

Yechish: Bilishimiz lozim: $y^` = (a^x)^` = a^x \ln a$

$$y^` = (a^{kx})^` = ka^{kx}\ln a. \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$a)\ \int 3^{5x}dx \ = \frac{3^{5x}}{5\ln 3} + C.\ \ b)\ \int 4^{2x}dx \ = \frac{4^{2x}}{2\ln 4} + C.$

$v)\ \int 5^{2x}dx \ = \dfrac{5^{2x}}{2\ln 5} + C.\ \ g)\ \int 19^{2x}dx \ = \dfrac{19^{2x}}{2\ln 19} + C.$

$d)\ \int 7^{2x}dx \ = \dfrac{7^{2x}}{7\ln 7} + C$

2.Quyidagi integrallarni hisoblang: $a)\ \int sin3xdx \qquad b) \int sin4xdx$

$$v)\ \int sin7xdx \quad g)\ \int sin22xdx \quad d)\ \int sin33xdx.$$

Yechish: Bilishimiz lozim: $y^` = (cosx)^` = -sinx,\ y^` = (cos2x)^` = -2sin2x$

$$\int sinxdx = -cosx + C.$$

$a)\ \int sin3xdx = -\frac{cos3x}{3} + C.\ \ b) \int sin4xdx = -\frac{cos4x}{4} +\ C.$

$v)\ \int sin7xdx = -\dfrac{cos7x}{7} + C.\ \ g) \int sin22xdx = -\dfrac{cos22x}{22} +\ C.$

$d)\ \int sin33xdx = -\dfrac{cos33x}{33} + C.$

3.Quyidagi integrallarni hisoblang: $a)\ \int cos2xdx \qquad b) \int cos3xdx$

$$v) \int cos5xdx \quad g)\ \int cos19xdx \quad d)\ \int cos21xdx.$$

Yechish: Bilishimiz lozim: $y^` = (sinx)^` = cosx \qquad y^` = (sin2x)^` = 2cos2x$

$$\int cosdx = sinx + C.$$

$a)\ \int cos2xdx = \frac{sin2x}{2} + C.\ \ b) \int cos3xdx = \frac{sin3x}{3} +\ C.$

$v)\ \int cos5xdx = \dfrac{sin5x}{5} + C.\ \ g) \int cos19xdx = \dfrac{sin19x}{19} +\ C.$

$d)\ \int cos21xdx = \frac{sin21x}{21} + C.$

4.Quyidagi integrallarni hisoblang: $\quad a)\ \int sin3xcos2xdx$

$b) \int sin5xcos3xdx \quad v) \int cos2xcos3xdx \quad g) \int sin7x\ sin3xdx$

$d)\ \int sin2xcosxdx.$

Yechish: Quyidagi trigonometrik formulalarni bilishimiz lozim:

1) $sinx\ cosy = \frac{1}{2}(\sin(x+y) + \sin(x-y)).$

2) $1)\ cosx\ cosy = \frac{1}{2}(\cos(x+y) + \cos(x-y)).$

3) $1)\ sinx\ siny = \frac{1}{2}(\cos(x-y) - \cos(x+y)).$

$$a) \int sin3xcos2xdx = \frac{1}{2}\int (sin5x + sinx)dx = \frac{1}{2}\left(\int sin5xdx + \int sinxdx\right) =$$

$$= \frac{1}{2}\frac{1}{5}(-cos5x) + \frac{1}{2}(-cosx) + C = -\frac{1}{10}cos5x - \frac{1}{2}cosx + C.$$

$$b) \int sin5xcos3xdx = \frac{1}{2}\int (sin8x + sin2x)dx = \frac{1}{2}\left(\int sin8xdx + \int sin2xdx\right)$$

$$=$$

$$= \frac{1}{2}\frac{1}{8}(-cos8x) + \frac{1}{2}\frac{1}{2}(-cos2x) + C = -\frac{1}{16}cos8x - \frac{1}{4}cos2x + C.$$

$$\frac{1}{2}\int (sin5x + sinx)dx = \frac{1}{2}\left(\int sin5xdx + \int sinxdx\right) =$$

$$v) \int cos2xcos3xdx = \frac{1}{2}\int (cos5x + \cos(-x))dx = \frac{1}{2}\left(\int cos5xdx + \int cosxdx\right)$$

$$=$$

$$= \frac{1}{2}\frac{1}{5}(sin5x) + \frac{1}{2}sinx + C = \frac{1}{10}sin5x + \frac{1}{2}sinx + C.$$

$$g) \int sin7xsin3xdx = \frac{1}{2}\int (cos4x + cos10x)dx = \frac{1}{2}\left(\int cos4xdx + \int cos10xdx\right) =$$

$$= \frac{1}{2}\frac{1}{4}sin4x + \frac{1}{2}\frac{1}{10}sin10x + C = \frac{1}{8}sin4x + \frac{1}{20}sin10x + C.$$

$$d)\ \int sin2xcosxdx = \frac{1}{2}\int (cosx + cos3x)dx = \frac{1}{2}\left(\int cosxdx + \int cos3xdx\right) =$$

$$= \frac{1}{2}sinx + \frac{1}{2}\frac{1}{3}sin3x + C = \frac{1}{2}sinx + \frac{1}{6}sin3x + C.$$

5.Quyidagi integrallarni hisoblang: $a) \int \frac{dx}{cos^2 3x}$ $b) \int \frac{dx}{sin^2 4x}$

$$v) \int \frac{dx}{cos^2 25x} \quad g) \int \frac{dx}{cos^2 4x} \quad d) \int \frac{dx}{sin^2 3x}$$

Yechish: Quyidagi trigonometrik formulalarni bilishimiz lozim:

$y` = (tgx)` = \frac{1}{cos^2x}$, $y` = (tg2x)` = \frac{2}{cos^2 2x}$,

$y` = (ctgx)` = -\frac{1}{sin^2x}$, $y` = (ctg2x)` = -\frac{2}{sin^2 2x}$.

$$\int \frac{dx}{cos^2x} = tgx + C; \quad \int \frac{dx}{sin^2x} = -ctgx + C.$$

$a) \int \frac{dx}{cos^2 3x} = \frac{1}{3}tg3x + C$. $b) \int \frac{dx}{sin^2 4x} = -\frac{1}{4}ctg4x + C$.

$v) \int \frac{dx}{cos^2 25x} = \frac{1}{25}tg25x + C$. $d) \int \frac{dx}{sin^2 3x} = -\frac{1}{3}ctg3x + C$.

$$g) \int \frac{dx}{cos^2 4x} = \frac{1}{4}tg4x + C.$$

6.Quyidagi integrallarni hisoblang: $a) \int \frac{dx}{1+9x^2}$ $b) \int \frac{dx}{1+4x^2}$ $b) \int \frac{dx}{1+16x^2}$

$g) \int \frac{dx}{1+49x^2}$ d) $\int \frac{dx}{1+169x^2}$.

Yechish: Bilishimiz lozim: $\int \frac{dx}{1+x^2} = arctgx + C$.

$a) \int \frac{dx}{1+9x^2} = \int \frac{dx}{1+(3x)^2} = \frac{1}{3}arctg3x + C.$

$b) \int \frac{dx}{1+4x^2} = \int \frac{dx}{1+(2x)^2} = \frac{1}{2}arctg2x + C.$

$v) \int \frac{dx}{1+16x^2} = \int \frac{dx}{1+(4x)^2} = \frac{1}{4}arctg4x + C.$

$g) \int \frac{dx}{1+49x^2} = \int \frac{dx}{1+(7x)^2} = \frac{1}{7}arctg7x + C.$

$d) \int \frac{dx}{1+169x^2} = \int \frac{dx}{1+(13x)^2} = \frac{1}{13}arctg13x + C$

7.Quyidagi integrallarni hisoblang: $a) \int \frac{dx}{\sqrt{1-9x^2}}$ $b) \int \frac{dx}{\sqrt{1-4x^2}}$ $v) \int \frac{dx}{\sqrt{1-16x^2}}$

$g) \int \frac{dx}{\sqrt{1-49x^2}}$ d) $\int \frac{dx}{\sqrt{1-169x^2}}$.

Yechish: Bilishimiz lozim: $\int \frac{dx}{1+x^2} = arcsinx + C$.

$a) \int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{1-(3x)^2}} = \frac{1}{3}arcsin3x + C.$

$b) \int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}} = \frac{1}{2} arcsin2x + C.$

$v) \int \frac{dx}{\sqrt{1-16x^2}} = \int \frac{dx}{\sqrt{1-(4x)^2}} = \frac{1}{4} arcsin4x + C.$

$g) \int \frac{dx}{\sqrt{1-49x^2}} = \int \frac{dx}{\sqrt{1-(7x)^2}} = \frac{1}{7} arcsin7x + C.$

$d) \int \frac{dx}{\sqrt{1-169x^2}} = \int \frac{dx}{\sqrt{1-(13x)^2}} = \frac{1}{13} arcsin13x + C.$

8.Quyidagi integrallarni hisoblang: $a) \int \frac{dx}{2x-1}$ $b) \int \frac{dx}{5x-2}$ $v) \int \frac{dx}{15x-21}$

$g) \int \frac{dx}{3x-1}$ d) $\int \frac{dx}{7x-3}$.

Yechish: Bilishimiz lozim: $\int \frac{dx}{x} = \ln(x) + C.$ $y` = (lnx)` = \frac{1}{x}, y` = (\ln(2x-1)` = \frac{2}{2x-1}.$

$a) \int \frac{dx}{2x-1} = \frac{1}{2} \ln|2x-1|$ $b) \int \frac{dx}{5x-1} = \frac{1}{5} \ln|5x-1|.$

$v) \int \frac{dx}{15x-1} = \frac{1}{15} \ln|15x-1|.$ $g) \int \frac{dx}{3x-1} = \frac{1}{3} \ln|3x-1|.$

$d) \int \frac{dx}{7x-1} = \frac{1}{7} \ln|7x-1|.$

9.Quyidagi integrallarni hisoblang: $a) \int \frac{x}{x+1} dx$ $b) \int \frac{dx}{x^2-7x+12}$

$v) \int \frac{dx}{x^2-3x-10}$ $g) \int \frac{dx}{x^2-4x+3}$ d) $\int \frac{dx}{4(x^2-4)}$.

Yechish.

$$a) \int \frac{x}{x+1} dx = \int \frac{x+1-1}{x+1} dx = \int \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right) dx =$$

$$= \int dx - \int \frac{1}{x+1} dx = x - \ln(x+1) + C.$$

$$b) \frac{dx}{x^2-7x+12} = \int \frac{dx}{(x-4)(x-3)} = \int \left(\frac{A}{x-4} + \frac{B}{x-3}\right) dx =$$

$$= \begin{cases} A(x-3) + B(x-4) = x*0+1 \\ Ax - 3A + Bx - 4B = x*0+1 \end{cases}$$

$$=> \begin{cases} x(A+B) + (-3A-4B) = x*0+1 \\ A+B=0\ A=-B,\ (3B-4B)=1,\ B=-1 \end{cases} => \begin{cases} A=1 \\ B=-1 \end{cases} =$$

$$\int\left(\frac{1}{x-4}-\frac{1}{x-3}\right)dx=\int\frac{1}{x-4}dx-\int\frac{1}{x-3}dx=\ln\left|\frac{x-4}{x-3}\right|.$$

ТРИГОНОМЕТРИЯ

НОВАЯ ШКОЛА

α, рад	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\alpha,°$	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt3}{2}$	1	$\frac{\sqrt3}{2}$	$\frac{\sqrt2}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt2}{2}$	$-\frac{\sqrt3}{2}$	-1	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt2}{2}$	$-\frac{1}{2}$	0
$\cos\alpha$	1	$\frac{\sqrt3}{2}$	$\frac{\sqrt2}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt2}{2}$	$-\frac{\sqrt3}{2}$	-1	$-\frac{\sqrt3}{2}$	$-\frac{\sqrt2}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt3}{2}$	1
$tg\alpha$	0	$\frac{1}{\sqrt3}$	1	$\sqrt3$	-	$-\sqrt3$	-1	$-\frac{1}{\sqrt3}$	0	$\frac{1}{\sqrt3}$	1	$\sqrt3$	-	$-\sqrt3$	-1	$-\frac{1}{\sqrt3}$	0
$ctg\alpha$	-	$\sqrt3$	1	$\frac{1}{\sqrt3}$	0	$-\frac{1}{\sqrt3}$	-1	$-\sqrt3$	-	$\sqrt3$	1	$\frac{1}{\sqrt3}$	0	$-\frac{1}{\sqrt3}$	-1	$-\sqrt3$	-

1. $\cos^2\alpha+\sin^2\alpha=1;\quad tg\alpha\cdot ctg\alpha=1;$
2. $tg\alpha=\frac{\sin\alpha}{\cos\alpha};\quad ctg\alpha=\frac{\cos\alpha}{\sin\alpha};$
3. $tg\alpha=\frac{1}{ctg\alpha};\quad ctg\alpha=\frac{1}{tg\alpha};$
4. $1+tg^2\alpha=\frac{1}{\cos^2\alpha};\quad 1+ctg^2\alpha=\frac{1}{\sin^2\alpha};$
5. $\sin(\alpha\pm\beta)=\sin\alpha\cdot\cos\beta\pm\cos\alpha\cdot\sin\beta;$
6. $\cos(\alpha\pm\beta)=\cos\alpha\cdot\cos\beta\mp\sin\alpha\cdot\sin\beta;$
7. $tg(\alpha\pm\beta)=\frac{tg\alpha\pm tg\beta}{1\mp tg\alpha\cdot tg\beta};$
8. $\sin2\alpha=2\sin\alpha\cdot\cos\alpha;$
9. $\cos2\alpha=\cos^2\alpha-\sin^2\alpha;$
10. $\cos2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha;$
11. $tg2\alpha=\frac{2tg\alpha}{1-tg^2\alpha};\quad ctg2\alpha=\frac{ctg^2\alpha-1}{2ctg\alpha};$
12. $\sin\alpha\pm\sin\beta=2\sin\frac{\alpha\pm\beta}{2}\cdot\cos\frac{\alpha\mp\beta}{2};$
13. $\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2};$
14. $\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2};$
15. $tg\alpha\pm tg\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cdot\cos\beta};$
16. $\sin^2\alpha=\frac{1-\cos2\alpha}{2};\quad \cos^2\alpha=\frac{1+\cos2\alpha}{2};$
17. $\sin\alpha\cdot\cos\beta=\frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta));$
18. $\cos\alpha\cdot\cos\beta=\frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta));$
19. $\sin\alpha\cdot\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$
20. $\left|\sin\frac{\alpha}{2}\right|=\sqrt{\frac{1-\cos\alpha}{2}};\quad \left|\cos\frac{\alpha}{2}\right|=\sqrt{\frac{1+\cos\alpha}{2}};$
21. $tg\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha};\quad ctg\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha}.$

6-amaliy mavzuga:

Integral ostidagi ifodani sodda qo‘shiluvchilar yig‘indisi ko‘rinishiga keltirib integrallashga yoyib integrallash usuli deyiladi.

Integrallarni toping:

1235.$\int\left(x^2+2x+\frac{1}{x}\right)dx=\int x^2dx+2\int xdx+\int\frac{1}{x}dx=\frac{x^3}{3}+2\cdot\frac{x^2}{2}+\ln|x|+C=\frac{x^3}{3}+2x+\ln|x|+C.$

1237.$\int\frac{x-2}{x^3}dx=\int\frac{x}{x^3}dx+\int\frac{2}{x^3}dx=\int x^{-2}\,dx+2\int x^{-3}dx=\frac{x^{-2+1}}{-2+1}+2\cdot\frac{x^{-3+1}}{-3+1}+C=-\frac{1}{x}-\frac{1}{x^2}+C.$

1239.$\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx=\int\sqrt{x}dx+\int\sqrt[3]{x}\,dx=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{4}{3}}}{\frac{4}{3}}+C=\frac{2x\sqrt{x}}{3}+\frac{3x\sqrt[3]{x}}{4}+C=x\left(\frac{2}{3}\sqrt{x}+\frac{3}{4}\sqrt[3]{x}\right)+C.$

1241. $\int\frac{\left(\sqrt{x}-1\right)^3}{x}dx=\int\frac{\left(\sqrt{x}\right)^3-3\left(\sqrt{x}\right)^2-3\sqrt{x}\cdot1-1}{x}dx=\int\frac{\sqrt{x^3}}{x}dx-3\int\frac{x}{x}dx-3\int\frac{\sqrt{x}}{x}dx-\int\frac{1}{x}dx=\int x^{\frac{3-2}{2}}dx-3\int dx-3\int x^{\frac{1-2}{2}}dx-\int\frac{dx}{x}=\frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}}-3x-3\cdot\frac{x^{\frac{-1+2}{2}}}{\frac{-1+2}{2}}-\ln|x|+C=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3x-3\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+\ln|x|+C=\frac{2\sqrt{x^3}}{3}-3x+6\sqrt{x}-\ln|x|+C.$

1243. $\int e^x\left(1-\frac{e^{-x}}{x^2}\right)dx=\int e^xdx-\int x^{-2}dx=e^x-\frac{x^{-2+1}}{-2+1}+C=e^x+\frac{1}{x}+C.$

1245.$\int\frac{cos2x}{cos^2x\cdot sin^2x}dx=\int\frac{cos^2x-sin^2x}{cos^2x\cdot sin^2x}dx=\int\frac{cos^2x}{cos^2x\cdot sin^2x}dx-\int\frac{sin^2x}{cos^2x\cdot sin^2x}dx=\int\frac{dx}{sin^2x}-\int\frac{dx}{cos^2x}=-ctgx-tgx+C.$

1247. $\int\frac{dx}{sin^2x\cdot cos^2x}=\int\frac{sin^2x+cos^2x}{sin^2x\cdot cos^2x}dx=\int\frac{sin^2x}{sin^2x\cdot cos^2x}dx+\int\frac{cos^2x}{sin^2x\cdot cos^2x}dx=\int\frac{dx}{cos^2x}+\int\frac{dx}{sin^2x}=tgx-ctgx+C.$

1249. $\int sin^2\frac{x}{2}=\int\left(\sqrt{\frac{1-cosx}{2}}\right)^2dx=\int\frac{1-cosx}{2}dx=\frac{1}{2}\int dx-\frac{1}{2}\int cosxdx=\frac{x}{2}-\frac{sinx}{2}+C.$

1251.$\int\left(\frac{2}{1+x^2}-\frac{3}{\sqrt{1-x^2}}\right)dx=2\int\frac{dx}{(1+x^2)}-3\int\frac{dx}{\sqrt{1-x^2}}=arctgx-3arcsinx+C.$

1253. $\int\frac{(x^2-1)^2}{x^3}dx=\int\frac{x^4-2x^2+1}{x^3}dx=\int\frac{x^4}{x^3}dx-2\int\frac{x^2}{x^3}dx+\int\frac{1}{x^3}dx=\int xdx-2\int\frac{dx}{x}+\int x^{-3}dx=\frac{x^2}{2}-2\ln|x|+\frac{x^{-3+1}}{-3+1}+C=\frac{x^2}{2}-\frac{x^{-2}}{2}-2\ln|x|+C=\frac{x^4-1}{2x^2}-2\ln|x|+C.$

1255. $\int\frac{x-2}{\sqrt{x^3}}dx=\int x^{1-\frac{3}{2}}dx-2\int x^{-\frac{3}{2}}dx=\int x^{-\frac{1}{2}}dx-2\int x^{-\frac{3}{2}}dx=\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-2\cdot\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}+C=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}-2\cdot\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}+C=2\sqrt{x}-\frac{4}{\sqrt{x}}+C=\frac{2x+4}{\sqrt{x}}+C=\frac{2(x-2)}{\sqrt{x}}+C.$

1257. $\int\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)dx=\int\frac{dx}{x}+\int x^{-2}dx+\int x^{-3}dx=\ln|x|+\frac{x^{-2+1}}{-2+1}+\frac{x^{-3+1}}{-3+1}+C=\ln|x|-\frac{1}{x}-\frac{1}{2x^2}+C.$

1259. $\int e^x\left(1+\frac{e^{-x}}{cos^2x}\right)dx=\int\left(e^x+\frac{1}{cos^2x}\right)dx=\int e^x\,dx+\int\frac{dx}{cos^2x}=e^x+tgx+C.$

1261. $\int\frac{1-sin^3x}{sin^2x}dx=\int\frac{dx}{sin^2x}-\int sinxdx=-ctgx+cosx+C.$

Bevosita integrallash va o‘rniga qo‘yish usullari:

1263. $\int cos3xdx=\frac{1}{3}sin3x+C.$

1265. $\int e^{-3x}dx=-\frac{1}{3}e^{-3x}+C.$

1267. $\int\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)dx=\int e^{\frac{x}{2}}dx+\int e^{-\frac{x}{2}}dx=2e^{\frac{x}{2}}-2e^{-\frac{x}{2}}+C=2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)+C.$

1269. $\int(3-2x)^4dx=-\frac{(3-2x)^5}{2\cdot5}+C=-\frac{(3-2x)^5}{10}+C.$

1271. $\int\frac{dx}{\sqrt{3-2x}}=\int\left(\sqrt{3-2x}\right)^{-\frac{1}{2}}dx=\frac{(3-2x)^{-\frac{1+2}{2}}}{2\left(\frac{1+2}{2}\right)}+C=-(3-2x)^{\frac{1}{2}}+C=-\sqrt{3-2x}+C.$

1273. $\int\frac{2x-5}{x^2-5x+7}dx=\frac{1}{2}\int\frac{(2x-5)^{`}d(2x-5)}{x^2-5x+7}=\ln(x^2-5x+7)+C.$

1275. $\int\frac{dx}{1-10x}=-\frac{1}{10}\ln|1-10x|+C.$

1277. $\int ctgxdx=\int\frac{cosxdx}{sinx}=\int\frac{dsinx}{sinx}=\ln|sinx|+C.$

1279. $\int \frac{cos2x}{sinx\ cosx} dx = \int \frac{cos^2x - sin^2x}{sinx\ cosx} dx = \int \frac{cos^2x}{sinx\ cosx} dx - \int \frac{sin^2x}{sinx\ cosx} dx =$
$\int \frac{cosx}{sinx} dx - \int \frac{sinx}{cosx} dx = \int \frac{dsinx}{sinx} + \int \frac{dcosx}{cosx} = \ln|sinx| + \ln|cosx| + C =$
$\ln|sinx\ cosx| + C = \frac{1}{2}\ln|sin2x| + C.$

1281. $\int \frac{cosx}{1+2sinx} dx = \frac{1}{2}\int \frac{d(1+2sinx)}{1+2sinx} = \frac{1}{2}\ln|1 + 2sinx|.$

1283. $\int sin^2x\ cosx\ dx = \int sin^2x dsinx = \frac{sin^3x}{3} + C.$

1285. $\int \frac{cosx\ dx}{sin^4x} = \int \frac{dsinx}{sin^4x} = -\frac{1}{3sin^3x} + C.$

6-amaliy mavzuga:

Integral ostidagi ifodani sodda qo‘shiluvchilar yig‘indisi ko‘rinishiga keltirib integrallashga yoyib integrallash usuli deyiladi.

Integrallarni toping:

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1237.$\int\frac{x-2}{x^3}dx=\int\frac{x}{x^3}dx+\int\frac{2}{x^3}dx=\int x^{-2}\,dx+2\int x^{-3}dx=\frac{x^{-2+1}}{-2+1}+2\cdot\frac{x^{-3+1}}{-3+1}+C=-\frac{1}{x}-\frac{1}{x^2}+C.$

1239.$\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx=\int\sqrt{x}dx+\int\sqrt[3]{x}\,dx=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{4}{3}}}{\frac{4}{3}}+C=\frac{2x\sqrt{x}}{3}+\frac{3x\sqrt[3]{x}}{4}+C=x\left(\frac{2}{3}\sqrt{x}+\frac{3}{4}\sqrt[3]{x}\right)+C.$

1241. $\int\frac{\left(\sqrt{x}-1\right)^3}{x}dx=\int\frac{\left(\sqrt{x}\right)^3-3\left(\sqrt{x}\right)^2-3\sqrt{x}\cdot1-1}{x}dx=\int\frac{\sqrt{x^3}}{x}dx-3\int\frac{x}{x}dx-3\int\frac{\sqrt{x}}{x}dx-\int\frac{1}{x}dx=\int x^{\frac{3-2}{2}}dx-3\int dx-3\int x^{\frac{1-2}{2}}dx-\int\frac{dx}{x}=\frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}}-3x-3\cdot\frac{x^{\frac{-1+2}{2}}}{\frac{-1+2}{2}}-\ln|x|+C=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3x-3\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+\ln|x|+C=\frac{2\sqrt{x^3}}{3}-3x+6\sqrt{x}-\ln|x|+C.$

1243. $\int e^x\left(1-\frac{e^{-x}}{x^2}\right)dx=\int e^xdx-\int x^{-2}dx=e^x-\frac{x^{-2+1}}{-2+1}+C=e^x+\frac{1}{x}+C.$

1245.$\int\frac{cos2x}{cos^2x\cdot sin^2x}dx=\int\frac{cos^2x-sin^2x}{cos^2x\cdot sin^2x}dx=\int\frac{cos^2x}{cos^2x\cdot sin^2x}dx-\int\frac{sin^2x}{cos^2x\cdot sin^2x}dx=\int\frac{dx}{sin^2x}-\int\frac{dx}{cos^2x}=-ctgx-tgx+C.$

1247. $\int\frac{dx}{sin^2x\cdot cos^2x}=\int\frac{sin^2x+cos^2x}{sin^2x\cdot cos^2x}dx=\int\frac{sin^2x}{sin^2x\cdot cos^2x}dx+\int\frac{cos^2x}{sin^2x\cdot cos^2x}dx=\int\frac{dx}{cos^2x}+\int\frac{dx}{sin^2x}=tgx-ctgx+C.$

1249. $\int sin^2\frac{x}{2}=\int\left(\sqrt{\frac{1-cosx}{2}}\right)^2dx=\int\frac{1-cosx}{2}dx=\frac{1}{2}\int dx-\frac{1}{2}\int cosxdx=\frac{x}{2}-\frac{sinx}{2}+C.$

1251.$\int\left(\frac{2}{1+x^2}-\frac{3}{\sqrt{1-x^2}}\right)dx=2\int\frac{dx}{(1+x^2)}-3\int\frac{dx}{\sqrt{1-x^2}}=arctgx-3arcsinx+C.$

1253. $\int \frac{(x^2-1)^2}{x^3}dx = \int \frac{x^4-2x^2+1}{x^3}dx = \int \frac{x^4}{x^3}dx - 2\int \frac{x^2}{x^3}dx + \int \frac{1}{x^3}dx = \int xdx - 2\int \frac{dx}{x} + \int x^{-3}dx = \frac{x^2}{2} - 2\ln|x| + \frac{x^{-3+1}}{-3+1} + C = \frac{x^2}{2} - \frac{x^{-2}}{2} - 2\ln|x| + C = \frac{x^4-1}{2x^2} - 2\ln|x| + C.$

1255. $\int \frac{x-2}{\sqrt{x^3}}dx = \int x^{1-\frac{3}{2}}dx - 2\int x^{-\frac{3}{2}}dx = \int x^{-\frac{1}{2}}dx - 2\int x^{-\frac{3}{2}}dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\cdot\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2\cdot\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + C = 2\sqrt{x} - \frac{4}{\sqrt{x}} + C = \frac{2x+4}{\sqrt{x}} + C = \frac{2(x-2)}{\sqrt{x}} + C.$

1257. $\int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)dx = \int \frac{dx}{x} + \int x^{-2}dx + \int x^{-3}dx = \ln|x| + \frac{x^{-2+1}}{-2+1} + \frac{x^{-3+1}}{-3+1} + C = \ln|x| - \frac{1}{x} - \frac{1}{2x^2} + C.$

1259. $\int e^x\left(1 + \frac{e^{-x}}{cos^2x}\right)dx = \int \left(e^x + \frac{1}{cos^2x}\right)dx = \int e^x dx + \int \frac{dx}{cos^2x} = e^x + tgx + C.$

1261. $\int \frac{1-sin^3x}{sin^2x}dx = \int \frac{dx}{sin^2x} - \int sinxdx = -ctgx + cosx + C.$

Bevosita integrallash va o‘rniga qo‘yish usullari:

1263. $\int cos3xdx = \frac{1}{3}sin3x + C.$

1265. $\int e^{-3x}dx = -\frac{1}{3}e^{-3x} + C.$

1267. $\int \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)dx = \int e^{\frac{x}{2}}dx + \int e^{-\frac{x}{2}}dx = 2e^{\frac{x}{2}} - 2e^{-\frac{x}{2}} + C = 2\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) + C.$

1269. $\int (3-2x)^4dx = -\frac{(3-2x)^5}{2\cdot 5} + C = -\frac{(3-2x)^5}{10} + C.$

1271. $\int \frac{dx}{\sqrt{3-2x}} = \int\left(\sqrt{3-2x}\right)^{-\frac{1}{2}}dx = \frac{(3-2x)^{-\frac{1+2}{2}}}{2\left(\frac{1+2}{2}\right)} + C = -(3-2x)^{\frac{1}{2}} + C = -\sqrt{3-2x} + C.$

1273. $\int \frac{2x-5}{x^2-5x+7}dx = \frac{1}{2}\int \frac{(2x-5)^{`}d(2x-5)}{x^2-5x+7} = \ln(x^2-5x+7) + C.$

1275. $\int \frac{dx}{1-10x} = -\frac{1}{10}\ln|1-10x| + C.$

1277. $\int ctgxdx = \int \frac{cosxdx}{sinx} = \int \frac{dsinx}{sinx} = \ln|sinx| + C.$

1279. $\int \frac{cos2x}{sinx\ cosx} dx = \int \frac{cos^2x - sin^2x}{sinx\ cosx} dx = \int \frac{cos^2x}{sinx\ cosx} dx - \int \frac{sin^2x}{sinx\ cosx} dx =$ $\int \frac{cosx}{sinx} dx - \int \frac{sinx}{cosx} dx = \int \frac{dsinx}{sinx} + \int \frac{dcosx}{cosx} = \ln|sinx| + \ln|cosx| + C =$ $\ln|sinx\ cosx| + C = \frac{1}{2}\ln|sin2x| + C.$

1281. $\int \frac{cosx}{1+2sinx} dx = \frac{1}{2}\int \frac{d(1+2sinx)}{1+2sinx} = \frac{1}{2}\ln|1 + 2sinx|.$

1283. $\int sin^2x\ cosx\ dx = \int sin^2x dsinx = \frac{sin^3x}{3} + C.$

1285. $\int \frac{cosx\ dx}{sin^4x} = \int \frac{dsinx}{sin^4x} = -\frac{1}{3sin^3x} + C.$

Mavzu: Ratsional funksiyalarni sodda ratsional kasrlarga ajratish. Ratsional funksiyalarni integrallash algoritmi.

Ikkita ko'phadning nisbati ratsional kasr deyiladi. $\frac{P_n(x)}{Q_m(x)}$. Bunda $n<m$ bo'lsa ratsional kasr to'g'ri ratsional kasr deyiladi, $n\geq m$ bo'lsa, ratsional kasr noto'g'ri ratsional kasr deyiladi. Bunday kasr suratini maxrajiga bo'lish bilan butun va kasr qismlarga ajratiladi. Bundagi kasr to'g'ri kasr bo'ladi. Agar ratsional kasrda maxraj ya'ni $Q_m(x)=1$ bo'lsa, kasr butun ratsional funksiyaga aylanadi. Buni integrallash yuqorida ko'rib o'tilgan.

Endi to'g'ri ratsional kasrni integrallashni ko'rib o'tamiz. Avval oddiy ratsional kasrlarni integrallashni ko'ramiz. Umumiy holda ratsional kasr oddiy kasrlarga ajratilib, so'ngra integrallanadi. Oddiy ratsional kasrlar (ba'zan elementar kasrlar deb ham yuritiladi) qo'yidagi ko'rinishda bo'ladi.

1. $\frac{A}{x-a}$ 2. $\frac{A}{(x-a)^k}$ bu yerda $k\geq 2$ butun musbat son.

3. $\frac{Ax+B}{x^2+px+q}$. Maxrajni ildizi kompleks sonlardan iborat, ya'ni $\frac{p^2}{4}-q\leq 0$

4. $\frac{Ax+B}{(x^2+px+q)^k}$. $k\geq 2$ bo'lgan butun musbat son.

(1)-(4) ko'rinishdagi kasrlar eng sodda ratsional kasrlardir.

Endi shu kasrlarni integrallashni ko'raylik.

1. $\int\frac{A}{x-a}dx = A\ln|x-a|+C \qquad (C=const)$

2. $\int\frac{A}{(x-a)^k}dx = A\int(x-a)^{-k}d(x-a) =$

$= A\frac{(x-a)^{-k+1}}{-k+1}+C = \frac{A}{(1-k)(x-a)^{k-1}}+C$

3. $\int\frac{Ax+B}{x^2+px+q}dx = \int\frac{\frac{A}{2}(2x+p)+(B-\frac{Ap}{2})}{x^2+px+q}dx = \frac{A}{2}\int\frac{2x+p}{x^2+px+q}dx+$

$+(B-\frac{Ap}{2})\int\frac{dx}{x^2+px+q} = \left[\begin{matrix} x^2+px+q=t & \int\frac{dt}{t}=\ln|t|+C \\ (2x+p)dx=dt & \end{matrix}\right] =$

$= \frac{A}{2}\ln|x^2+px+q|+(B-\frac{Ap}{2})\int\frac{dx}{(x+p/2)^2+(q-p^2/4)} =$

$= \frac{A}{2}\ln|x^2+px+q|+\frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C$

4. $\int\frac{Ax+B}{(x^2+px+q)^k}dx = \int\frac{A/2(2x+p)+B-Ap/2)}{(x^2+px+q)^k}dx =$

$= \frac{A}{2}\int\frac{2x+p}{(x^2+px+q)^k}dx+(B-\frac{Ap}{2})\int\frac{dx}{(x^2+px+q)^k}.$

1. $\int \frac{2x+3}{x^2+3x-10}dx =$ **integralni hisoblang.**

Yechish: Bu integralni quyidagi uchta usulda yechimini topamiz;

1. $\int \frac{2x+3}{x^2+3x-10}dx = \int \frac{dx}{(x+5)} + \int \frac{dx}{(x-2)} = \ln(x-5) + \ln(x-2) + C =$

$= \ln\big((x+5)(x-2)\big) + C = \ln(x^2+3x-10) + C.$

$$\frac{2x+3}{x^2+3x-10} = \frac{2x+3}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$$

$2x+3 = Ax - 2A + Bx + 5B = (A+B)x + (-2A+5B)$

$$\begin{cases} A+B=2 \\ -2A+5B=3 \end{cases} => \begin{cases} A=2-B \\ -2B+5(2-B)=3 \end{cases} => \begin{cases} A=2-B \\ -7B=-7 \end{cases}$$

Bundan A=1, B=1 hosil bo‘ladi.

2.$\int \frac{2x+3}{x^2+3x-10}dx = \int \frac{2\left(t-\frac{3}{2}\right)+3}{t^2-\frac{49}{4}}dt = \int \frac{2t}{t^2-\frac{49}{4}}dt = \int \frac{d\left(t^2-\frac{49}{4}\right)}{t^2+\frac{49}{4}} = \ln\left(t^2-\frac{49}{4}\right) + C =$

$= \ln(x^2+3x-10) + C.$

$x^2+3x-10 = \left(x+\frac{3}{2}\right)^2 - \frac{49}{4}$ $deb\ olib, x+3x=t, dx=dt.$

3. $\int \frac{2x+3}{x^2+3x-10}dx = \int \frac{d(x^2+3x-10)}{x^2+3x-10} = |asos: d(x^2+3x-10) = (2x+3)dx|$

$= \ln(x^2+3x-10) + C.$

$Javob$: $\ln(x^2+3x-10) + C.$

2. $\int \frac{\boldsymbol{x^5dx}}{\boldsymbol{x^2+5x+6}}$ ***noto‘g‘ri kasrli integralni hisoblamiz***.

Yechish: Kasrni suratini maxrajiga bo‘lish orqali amalga oshiramiz.

$$\int \frac{x^5dx}{x^2+5x+6} = \int \left(x^3 - 5x^2 + 19x - 65 + \frac{439x+390}{x^2+5x+6}\right)dx =$$

$$= \frac{x^4}{4} - \frac{5x^3}{3} + \frac{19x^2}{2} - 65x + \int \left(\frac{927}{x+3} - \frac{488}{x+2}\right)dx =$$

$$= \frac{x^4}{4} - \frac{5x^3}{3} + \frac{19x^2}{2} - 65x + \ln\frac{(x+3)^{927}}{(x+2)^{488}} + C.$$

Javob: $\frac{x^4}{4} - \frac{5x^3}{3} + \frac{19x^2}{2} - 65x + \ln\frac{(x+3)^{927}}{(x+2)^{488}} + C.$

3. $\int \frac{\boldsymbol{2x^2+5x+5}}{\boldsymbol{(x^2-1)(x+2)}}$ ***dx integralni hisoblang***.

Yechish: $(x^2-1)(x+2) = (x-1)(x+1)(x+2).$

$$\frac{2x^2+5x+5}{(x^2-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{(A+B)x^2 + (3A+B)x + (2A-2B-C}{(x^2-1)(x+2)}$$

$$\begin{cases} A+B=2 \\ 3A-B=5 \\ 2A-2B-C=5 \end{cases} => A=2, B=-1, C=1.$$

$$\int \frac{2x^2+5x+5}{(x^2-1)(x+2)}dx = \int \left(\frac{2}{x-1} - \frac{1}{x+1} + \frac{1}{x+2}\right)dx =$$

$$= 2\ln|x-1| - \ln|x+1| + \ln|x+2| + C = \ln\left|\frac{(x-1)^2(x+2)}{(x+1)}\right| + C.$$

Javob: $\ln\left|\frac{(x-1)^2(x+2)}{(x+1)}\right| + C.$

4. $\int \frac{dx}{(x+1)(x+7}$ integralni hisoblang.

Yechish: A va B noma’lum koeffitsientlarni topamiz.

$$\frac{1}{(x+1)(x+7)} = \frac{A}{x+1} + \frac{B}{x+7} = \frac{Ax+7A+Bx+B}{(x+1)(x+7)}$$

$$1 = (A+B)x + (7A+B)$$

$$\begin{cases} A+B=0 \\ 7A+B=1 \end{cases} => A = \frac{1}{6}, B = -\frac{1}{6} \ ga\ ega\ bo`lamiz.$$

$$\int \frac{dx}{(x+1)(x+7} = \frac{1}{6}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{dx}{x+7} = \frac{1}{6}\ln|x+1| - \frac{1}{6}\ln|x+7| + C =$$

$$= \frac{1}{6}\ln\left|\frac{x+1}{x+7}\right| + C.$$

Javob: $\frac{1}{6}\ln\left|\frac{x+1}{x+7}\right| + C.$

5. $\int \frac{\mathbf{3x^2-5x+1}}{\mathbf{(x-2)^2(x+2)^2}}$ *integralni hisoblang*.

Yechish: To‘g‘ri kasrni sodda kasrlar yig‘indis ko‘rinishida yozamiz:

$$\frac{3x^2-5x+1}{(x-2)^2(x+2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$$3x^3 - 5x + 1 = A(x+2)^2(x-2) + B(x+2)^2 + C(x-2)^2(x+2) + D(x-2)^2$$

$$\begin{cases} x=2\ da\ 15=16B \\ x=-2\ da\ 13=16D \\ x^3\ da\ 3=A+C \\ x^2\ da\ 0=2A+B-2C+D \end{cases} => B = \frac{15}{16}, D = \frac{13}{16}, A = \frac{47}{8}, C = -\frac{23}{8}.$$

$$\int \frac{3x^2-5x+1}{(x-2)^2(x+2)^2} = \int \left(\frac{\frac{47}{8}}{x-2} + \frac{\frac{15}{16}}{(x-2)^2} - \frac{\frac{23}{8}}{x+2} + \frac{\frac{13}{16}}{(x+2)^2}\right)dx = \frac{47}{8}\ln|x-2| -$$

$$-\frac{23}{8}\ln|x+2| - \frac{15}{16(x-2)} - \frac{13}{16(x+2)} + C.$$

Javob: $\frac{47}{8}\ln|x-2|\frac{23}{8}\ln|x+2| - \frac{15}{16(x-2)} - \frac{13}{16(x+2)} + C$

Integrallarni hisoblang.

$$\int \frac{A}{x-a}dx = A\ln|x-a| + C.$$

1431. $\int \frac{x^3}{x-2}dx = \int\left(x^2+2x+4+\frac{8}{x-2}\right)dx = \frac{x^3}{3}+x^2+4x+8\ln|x-2|+C.$

1433. $\int \frac{x^5}{x^3+a^3}dx = \int\left(x^2+\frac{a^3x^2}{x^3+a^3}\right)dx = \int x^2dx + a^3\int\frac{x^2}{x^3+a^3}dx =$

$$= \frac{x^3}{3}+\frac{a^3}{3}\int\frac{d(x^3+a^3)}{x^3+a^3}+C = \frac{x^3}{3}+\frac{a^3}{3}\ln|x^3+a^3|+C.$$

1435. $\int \frac{2x+7}{x^2+x-2}dx = \int\frac{2x+7}{(x+2)(x-1)}dx = \int\frac{A}{x-1}dx+\int\frac{B}{x+2}dx = 3\int\frac{dx}{x-1}$

$$-\int\frac{dx}{x+2} = 3\ln|x-1| - \ln|x+2| + C = \ln\left|\frac{(x-1)^3}{x+2}\right| + C.$$

1437. $\int \frac{(x+1)^3}{x^2-x}dx = \int\frac{x^3+3x^2+3x+3}{x^2-x}dx = \int\left(x+4+\frac{7x+1}{x(x-1)}\right) =$

$$\int xdx + 4\int dx + \int\frac{7x+1}{(x-1)x} = \frac{x^2}{2}+4x+8\int\frac{dx}{x-1}-\int\frac{dx}{x} =$$

$$= \frac{x^2}{2}+4x+(8\ln|x-1|-\ln|x|)+C = \frac{x^2}{2}+4x+\ln\left|\frac{(x-1)^8}{x}\right|+C.$$

1441. $\int \frac{5x-1}{x^3-3x-2}dx = \int\frac{dx}{x-2}-\int\frac{dx}{x+1}+\int\frac{2dx}{(x+1)^2} = \ln|x-2|-\ln|x+1|-\frac{2}{x+1}+C.$

$$\frac{5x-1}{x^3-3x-2} = \frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\frac{5x-1}{x^3-3x-2} = \frac{A(x+1)(x+1)^2+B(x-2)(x+1)^2+C(x-2)(x+1)}{x^3-3x-2}$$

$$A = 1,\quad B = -1,\qquad C = 2.$$

1443. $\int \frac{4x-2{,}4}{x^2-0{,}2x+0{,}17}dx = \int\frac{4x-\frac{12}{5}}{x^2-\frac{1}{5}x+\frac{17}{100}}dx = \int\frac{\frac{20x-12}{5}}{\frac{100x^2-20x+17}{100}}dx = \int\frac{(20x-12)\times 20}{100x^2-20x+17}dx =$

$$\int\frac{400x-240}{100x^2-20x+17}dx = \int\frac{2(200x-20)-200}{100x^2-20x+17}dx = \int\frac{2(200x-20)}{100x^2-20x+17}dx -$$

$$\int\frac{200}{100x^2-20x+17}dx = 2\int\frac{d(100x^2-20x+17)}{100x^2-20x+17} - \frac{200}{100}\int\frac{dx}{\left(x^2-\frac{1}{5}x+\frac{1}{100}+\frac{4}{25}\right)}$$

$=2\ln|100x^2-20x+17|-5arctg\left(\frac{10x-1}{4}\right).$

$$\frac{200}{100}\int\frac{dx}{\left(x^2-\frac{1}{5}x+\frac{1}{100}+\frac{4}{25}\right)}=2\int\frac{dx}{\left(\left(x-\frac{1}{10}\right)^2+\frac{4}{25}\right)}=2\int\frac{dx}{t^2+\left(\frac{2}{5}\right)^2}=$$

$$=2\cdot\frac{1}{\frac{2}{5}}arctg\left(\frac{t}{\frac{2}{5}}\right)=5arctg\left(\frac{x-\frac{1}{10}}{\frac{2}{5}}\right)=5arctg\left(\frac{\frac{10x-1}{10}}{\frac{2}{5}}\right)=5arctg\left(\frac{10x-1}{4}\right).$$

$x-\frac{1}{10}=t$ *deb olsak*.

Javob: $2\ln|100x^2-20x+17|-5arctg\left(\frac{10x-1}{4}\right).$

7-amaliy mavzuga: Ikki o‘zgaruvchili ratsional funksiyalar. Trigonometrik funksiyalarni integrallash.

ТРИГОНОМЕТРИЯ

НОВАЯ ШКОЛА

α, рад	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\alpha,°$	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$tg\alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$ctg\alpha$	-	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	-	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	-

1. $\cos^2\alpha + \sin^2\alpha = 1; \quad tg\alpha \cdot ctg\alpha = 1;$
2. $tg\alpha = \frac{\sin\alpha}{\cos\alpha}; \quad ctg\alpha = \frac{\cos\alpha}{\sin\alpha};$
3. $tg\alpha = \frac{1}{ctg\alpha}; \quad ctg\alpha = \frac{1}{tg\alpha};$
4. $1 + tg^2\alpha = \frac{1}{\cos^2\alpha}; \quad 1 + ctg^2\alpha = \frac{1}{\sin^2\alpha};$
5. $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta;$
6. $\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta;$
7. $tg(\alpha \pm \beta) = \frac{tg\alpha \pm tg\beta}{1 \mp tg\alpha \cdot tg\beta};$
8. $\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha;$
9. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha;$
10. $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha;$
11. $tg 2\alpha = \frac{2tg\alpha}{1 - tg^2\alpha}; \quad ctg 2\alpha = \frac{ctg^2\alpha - 1}{2ctg\alpha};$
12. $\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha \pm \beta}{2} \cdot \cos\frac{\alpha \mp \beta}{2};$
13. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2} \cdot \cos\frac{\alpha - \beta}{2};$
14. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha + \beta}{2} \cdot \sin\frac{\alpha - \beta}{2};$
15. $tg\alpha \pm tg\beta = \frac{\sin(\alpha \pm \beta)}{\cos\alpha \cdot \cos\beta};$
16. $\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}; \quad \cos^2\alpha = \frac{1 + \cos 2\alpha}{2};$
17. $\sin\alpha \cdot \cos\beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta));$
18. $\cos\alpha \cdot \cos\beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta));$
19. $\sin\alpha \cdot \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$
20. $\left|\sin\frac{\alpha}{2}\right| = \sqrt{\frac{1 - \cos\alpha}{2}}; \quad \left|\cos\frac{\alpha}{2}\right| = \sqrt{\frac{1 + \cos\alpha}{2}};$
21. $tg\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}; \quad ctg\frac{\alpha}{2} = \frac{\sin\alpha}{1 - \cos\alpha}.$

1-misol: $\int cos^5 x sinx dx = \left| t = cosx, dx = \frac{1}{t^`} dt, t^` = -sinx \right| =$

$$= \int t^5 \, sinx \, \left(-\frac{1}{sinx}\right) dt = -\int t^5 dt = -\frac{t^6}{6} + C = -\frac{cos^6 x}{6} + C.$$

Javob: $-\frac{\boldsymbol{cos^6 x}}{\boldsymbol{6}} + \boldsymbol{C}.$

2-misol: $\int sin^3 x \, dx = \int sin^2 x \, sinx \, dx = \int (1 - cos^2 x) sinx dx =$

$$= \left| t = cosx, dx = \frac{1}{t^`} dt, t^` = -sinx \right| = \int (1 - t^2) sinx \, \left(-\frac{1}{sinx}\right) dt =$$

$$= -\int (1 - t^2) dt = \int t^2 dt - \int dt = \frac{t^3}{3} - t + C = \frac{t^3 - 3t}{3} + C =$$

$= \frac{cos^3 x - 3cosx}{3} + C.$ ***Javob:*** $\frac{\boldsymbol{cos^3 x - 3cosx}}{\boldsymbol{3}} + \boldsymbol{C}.$

3-misol: $\int cos^2 4x \, dx = \int \left(\sqrt{\frac{1+\cos 2(4x)}{2}}\right)^2 dx = \int \left(\frac{1+cos8x}{2}\right) dx =$

$$= \int \left(\frac{1}{2} - \frac{cos8x}{2}\right) dx = \frac{x}{2} - \frac{sin8x}{8 \cdot 2} + C = \frac{x}{2} - \frac{sin8x}{16} + C.$$

Javob: $\frac{\boldsymbol{x}}{\boldsymbol{2}} - \frac{\boldsymbol{sin8x}}{\boldsymbol{16}} + \boldsymbol{C}.$

4-misol: $\int \frac{sinx}{cos^3 x} \, dx = \left| t = cosx, \frac{dt}{-1} = \frac{-sinx}{-1} dx, -dt = sinx dx \right| =$

$$= \int \frac{-dt}{(t)^3} = -\int t^{-3} \, dt = -\frac{t^{-2}}{-2} + C = \frac{1}{2t^2} + C = \frac{1}{2cos^2 x} + C.$$

Javob: $\frac{\boldsymbol{1}}{\boldsymbol{2cos^2 x}} + \boldsymbol{C}.$

Integral ostidagi ifodani sodda qo‘shiluvchilar yig‘indisi ko‘rinishiga keltirib integrallashga yoyib integrallash usuli deyiladi.

Integrallarni toping:

1245. $\int \frac{cos2x}{cos^2 x \cdot sin^2 x} dx = \int \frac{cos^2 x - sin^2 x}{cos^2 x \cdot sin^2 x} dx = \int \frac{cos^2 x}{cos^2 x \cdot sin^2 x} dx - \int \frac{sin^2 x}{cos^2 x \cdot sin^2 x} dx =$ $\int \frac{dx}{sin^2 x} - \int \frac{dx}{cos^2 x} = -ctgx - tgx + C.$

1247. $\int \frac{dx}{sin^2 x \cdot cos^2 x} = \int \frac{sin^2 x + cos^2 x}{sin^2 x \cdot cos^2 x} dx = \int \frac{sin^2 x}{sin^2 x \cdot cos^2 x} dx + \int \frac{cos^2 x}{sin^2 x \cdot cos^2 x} dx =$ $\int \frac{dx}{cos^2 x} + \int \frac{dx}{sin^2 x} = tgx - ctgx + C.$

1249. $\int sin^2\frac{x}{2}=\int\left(\sqrt{\frac{1-cosx}{2}}\right)^2 dx=\int\frac{1-cosx}{2}dx=\frac{1}{2}\int dx-\frac{1}{2}\int cosxdx=\frac{x}{2}-\frac{sinx}{2}+C.$

1259.$\int e^x\left(1+\frac{e^{-x}}{cos^2x}\right)dx=\int\left(e^x+\frac{1}{cos^2x}\right)dx=\int e^x\,dx+\int\frac{dx}{cos^2x}=e^x+tgx+C.$

1261. $\int\frac{1-sin^3x}{sin^2x}dx=\int\frac{dx}{sin^2x}-\int sinxdx=-ctgx+cosx+C.$

Bevosita integrallash va o‘rniga qo‘yish usullari:

1263. $\int cos3xdx=\frac{1}{3}sin3x+C.$

1277. $\int ctgxdx=\int\frac{cosxdx}{sinx}=\int\frac{dsinx}{sinx}=\ln|sinx|+C.$

1279.$\int\frac{cos2x}{sinx\,cosx}dx=\int\frac{cos^2x-sin^2x}{sinx\,cosx}dx=\int\frac{cos^2x}{sinx\,cosx}dx-\int\frac{sin^2x}{sinx\,cosx}dx=$
$\int\frac{cosx}{sinx}dx-\int\frac{sinx}{cosx}dx=\int\frac{dsinx}{sinx}+\int\frac{dcosx}{cosx}=ln|sinx|+ln|cosx|+C=$
$ln|sinx\,cosx|+C=\frac{1}{2}ln|sin2x|+C.$

1281. $\int\frac{cosx}{1+2sinx}dx=\frac{1}{2}\int\frac{d(1+2sinx)}{1+2sinx}=\frac{1}{2}\ln|1+2sinx|.$

1283. $\int sin^2x\,cosx\,dx=\int sin^2xdsinx=\frac{sin^3x}{3}+C.$

1285. $\int\frac{cosx\,dx}{sin^4x}=\int\frac{dsinx}{sin^4x}=-\frac{1}{3sin^3x}+C.$

1391. $\int(1+2cosx)^2dx=\int(1+4cosx+4cos^2x)dx=\int dx+4\int cosxdx$
$+4\int cos^2xdx=x+4sinx+4\int\left(\frac{1+cos2x}{2}\right)dx=x+4sinx+\frac{4}{2}x+$
$\frac{4}{2\cdot 2}sin2x+C=3x+4sinx+sin2x+C.$

1393. $\int cos^4xdx = \int (cos^2x)^2\,dx = \int \left(\frac{1+cos2x}{2}\right)^2 dx = \int \frac{(1+cos2x)^2}{4} dx =$

$\int \frac{1+2cos2x+cos^2 2x}{4} dx = \frac{1}{4}\int (1+2cos2x+cos^2 2x)\,dx = \frac{1}{4}\int dx +$

$\frac{2}{4}\int cos2xdx + \frac{1}{4}\int cos^2 2xdx = \frac{x}{4} + \frac{sin2x}{2\cdot 2} + \frac{1}{4}\int \left(\frac{1+cos4x}{2}\right) dx = \frac{x}{4} +$

$\frac{sin2x}{4} + \frac{1}{4}\cdot\frac{1}{2}\int dx + \frac{1}{4}\cdot\frac{1}{2}\int cos4xdx = \frac{x}{4} + \frac{sin2x}{4} + \frac{x}{8} + \frac{1}{8}\cdot\frac{1}{4}sin4x + C =$

$= \frac{3x}{8} + \frac{sin2x}{4} + \frac{sin4x}{32} + C.$

1395. $\int sin^4x\,cos^4xdx = \int (sinx\,cosx)^4\,dx = \int \left(\frac{sin2x}{2}\right)^4 dx = \frac{1}{16}\int (sin^2 2x)^2\,dx =$

$\frac{1}{16}\int \left(\frac{1-cos4x}{2}\right)^2 dx = \frac{1}{16\cdot 4}\int (1-cos4x)^2 dx = \frac{1}{64}\int (1-2cos4x+cos^2 4x)dx =$

$\frac{1}{64}\left(\int dx - 2\int cos4x + \int \left(\frac{1+cos8x}{2}\right) dx\right) = \frac{1}{64}\left(x - \frac{2\,sin4x}{4} + \frac{1}{2}\int dx + \frac{1}{2}\int cos8xdx\right) =$

$\frac{1}{64}\left(x - \frac{sin4x}{2} + \frac{x}{2} + \frac{sin8x}{16}\right) = \frac{3x}{128} - \frac{sin4x}{128} + \frac{sin8x}{1024} + C.$

1397. $\int sin^5xdx = \int sin^4x\,sinx\,dx = \left|t = cosx, dx = \frac{1}{t^`}dt, t^` = -sinx\right| =$

$\int sin^4x\,sinx\cdot\frac{1}{-sinx}dt = \int -(sin^2x)^2 dt = \int -(1-cos^2x)^2 dt =$

$\int -(1-2cos^2x+cos^4x)dt = \int -(1-2t^2+t^4)\,dt = \int (-1+2t^2-t^4)dt =$

$-t + \frac{2t^3}{3} - \frac{t^5}{5} = -cosx + \frac{2cos^3x}{3} - \frac{cos^5x}{5} + C.$

1399. $\int sin^3xcos^3\,xdx = \left|t = sinx, dx = \frac{1}{t^`}dt, t^` = cosx\right| = \int sin^3xcos^3x\frac{1}{cosx}dt =$

$\int sin^3xcos^2xdt = \int sin^3x(1-sin^2x)\,dt = \int (sin^3x - sin^5x)dt = \int (t^3-t^5)dt =$

$\frac{t^4}{4} - \frac{t^6}{6} = \frac{sin^4x}{4} - \frac{sin^6x}{6} + C.$

1401. $\int (1+2cosx)^3\,dx = \int (1+6cosx+12cos^2x+8cos^3x)\,dx =$

$\int dx + 6\int cosxdx + 12\int cos^2xdx + 8\int cos^3xdx = x + 6sinx +$

$12\int \left(\frac{1-cos2x}{2}\right)dx + 8\int cos^2x\,cosxdx = \left|t = sinx, dx = \frac{1}{t`}dt, t` = cosx\right| =$

$x + 6sinx + \frac{12x}{2} - \frac{12sin2x}{2\cdot 2} + 8\int cos^2x\,cosx\cdot\frac{1}{cosx}dt = x + 6sinx + 6x$

$-3sin2x + 8\int cos^2xdt = 7x + 6sinx - 3sin2x + 8sinx - \frac{8sin^3x}{3} =$

$= 7x + 14sinx - 3sin2x - \frac{8sin^3x}{3} + C.$

Oxirgi integralni quyidagicha yechimi topiladi.

$8\int cos^2xdt = 8\int (1-sin^2x)\,dt = 8\int (1-t^2)\,dt = 8\left(\int dt - \int t^2dt\right) =$

$8\left(t - \frac{t^3}{3}\right) = 8\left(sinx - \frac{sin^3x}{3}\right) = 8sinx - \frac{8sin^3x}{3} + C.$

1403. $\int \frac{sin^3x\,dx}{cos^2x} = \left|t = cosx, dx = \frac{1}{t`}dt, t` = -sinx\right| = \int \frac{sin^3x}{cos^2x}\cdot\frac{1}{(-sinx)}dt =$

$\int -\frac{sin^2x}{cos^2x}dt = -\int \frac{(1-cos^2x)}{cos^2x}dt = -\int \left(\frac{1-t^2}{t^2}\right)dt = -\int t^{-2}\,dt + \int dt =$

$-\frac{t^{-2+1}}{-2+1} + t = \frac{1}{t} + t = \frac{1}{cosx} + cosx + C.$

1405. $\int \frac{dx}{sinx} = \int \left(\frac{1}{sinx}\cdot\frac{sinx}{sinx}\right)dx = \int \frac{sinx}{sin^2x}dx = \int \left(\frac{sinx}{1-cos^2x}\right)dx =$

$\left|t = cosx, dx = \frac{1}{t`}dt, t` = -sinx\right| = -\int \frac{dt}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| =$

$\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right| = \frac{1}{2}\ln\left|\frac{1-cosx}{1+cosx}\right| + C = \ln\left|\sqrt{\frac{1-cosx}{1+cosx}}\right| + C = \ln\left|tg\frac{x}{2}\right| + C.$

1407. $\int \frac{cosx+sinx}{sin2x}dx = \int \left(\frac{cosx}{sin2x} + \frac{sinx}{sin2x}\right)dx = \int \left(\frac{cosx}{2sinxcosx} + \frac{sinx}{2sinxcosx}\right)dx$

$$=\int\left(\frac{1}{2sinx}+\frac{1}{2cosx}\right)dx=\frac{1}{2}\left(\int\frac{dx}{sinx}+\int\frac{dx}{cosx}\right)=\frac{1}{2}\left(\int\left(\frac{sinx}{sin^2x}dx\right)+\int\left(\frac{cosx}{cos^2x}dx\right)\right)=$$

$$\frac{1}{2}\left(\frac{1}{2}\ln\left|\frac{1-cosx}{1+cosx}\right|+\frac{1}{2}\ln\left|\frac{1+sinx}{1-sinx}\right|\right)=\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+\frac{1}{4}\ln\left|\frac{1+sinx}{1-sinx}\right|+C$$

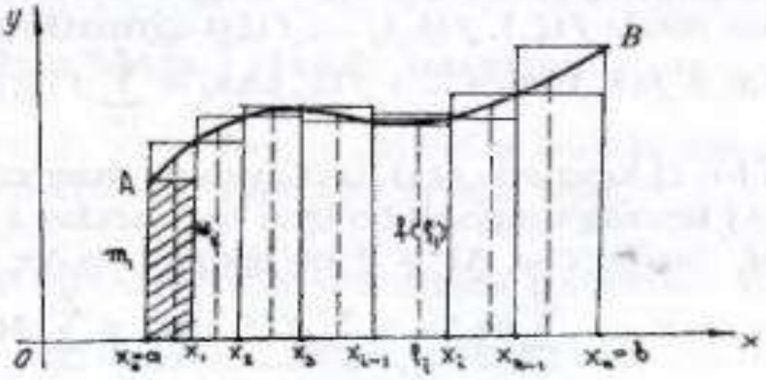
1) $\int\left(\frac{sinx}{sin^2x}dx\right)=\int\frac{sinx}{1-cos^2x}dx=\left|t=cosx,dx=\frac{1}{t`}dt,t`=-sinx\right|=$

$$\int\frac{sinx}{1-cos^2x}\cdot\frac{1}{(-sinx)}dt=-\int\frac{dt}{1-t^2}=\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|=\frac{1}{2}\ln\left|\frac{1-cosx}{1+cosx}\right|+C.$$

2) $\int\left(\frac{cosx}{cos^2x}dx\right)=\int\frac{cosx}{1-sin^2x}dx=\left|t=sinx,dx=\frac{1}{t`}dt,t`=cosx\right|=\int\frac{cosx}{1-sin^2x}\cdot\frac{1}{cosx}dt=$

$$\int\frac{1}{1-sin^2x}dt=\int\frac{dt}{1-t^2}=\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|=\frac{1}{2}\ln\left|\frac{1+sinx}{1-sinx}\right|+C.$$

$[a; b]$ kesmada $y = f(x)$ uzluksiz funksiya berilgan bo'lsin (119-chizma). Berilgan $y = f(x)$ funksiya grafigi, abssissalar o'qi, $x = a$ va $x = b$ vertikal to'g'ri chiziqlar bilan chegaralangan $aABb$ tekis figura *egri chiziqli trapetsiya* deyiladi. Shu egri chiziqli trapetsiya yuzini topamiz. Buning uchun $y = f(x)$ funksiyaning kesmadagi eng katta va eng kichik qiymatlarini mos ravishda M va m bilan belgilaymiz. $[a; b]$ kesmani $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, ..., n$ nuqtalar bilan n ta



kesmachalarga ajratamiz, bunda $x_0 < x_1 < x_2 < ... < x_n$ deb hisoblaymiz va $x_1 - x_0 = \Delta x_1, ..., x_2 - x_1 = \Delta x_2, x_n - x_{n-1} = \Delta x_n$ deb faraz qilamiz, so'ngra $f(x)$ funksiyaning eng kichik va eng katta qiymatlarini

$[x_0; x_1]$ kesmada m_1 va M_1 bilan,

$[x_1; x_2]$ kesmada m_2 va M_2 bilan,

. . . .

$[x_{n-1}; x_n]$ kesmada m_n va M_n bilan belgilaymiz.

Endi quyidagi yig'indilarni tuzamiz:

$$\underline{s_n} = m_1\Delta x_1 + m_2\Delta x_2 + ... + m_n\Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i,$$

$$\overline{s_n} = M_1\Delta x_1 + M_2\Delta x_2 + ... + M_n\Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i.$$

Bu yig'indilar *integral yig'indilar* deyilib, mos ravishda ichki va tashqi chizilgan zinasimon shaklni siniq chiziq bilan chegaralangan yuziga teng bo'ladi. Bundan esa $\underline{s_n} \le S_{aABb} \le \overline{s_n}$ tengsizlik o'rinli bo'ladi. Agar $[a; b]$ kesmalarni yana ham kichiklashtirib bo'laklarga ajratsak, n yetarli darajada katta bo'lganda $\underline{s_n}$ va $\overline{s_n}$ lar bir-biridan kam farq qiladi va egri chiziqli trapetsiyaning yuzini aniqlaydi.

Ta'rif. Aytaylik, $y = f(x)$ $x \in [a; b]$ manfiy bo'lmagan, uzluksiz funksiya bo'lsin. Bu holda, agar $\{\underline{s_n}\}$ va $\{\overline{s_n}\}$ ketma-ketliklar limitlari mavjud bo'lib, bir-biriga teng bo'lsa, limitning qiymati *egri chiziqli trapetsiyaning yuzi* deyiladi.

Endi $[x_0; x_1]$, $[x_1; x_2]$, ..., $[x_{n-1}; x_n]$ kesmalarning har birida bittadan nuqta olamiz. Bu nuqtalarni $\xi_1, \xi_2, ..., \xi_n$ bilan belgilaymiz. Bu nuqtalarning har birida $f(\xi_1), f(\xi_2), ..., f(\xi_n)$ qiymatlarni hisoblaymiz va $s_n = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ yig'indini tuzamiz.

Bu yig'indi $[a; b]$ kesmada $f(x)$ funksiyaning *integral yig'indisi* deb ataladi. $[x_{i-1}; x_i]$ kesmaga tegishli bo'lgan har qanday ξ_i nuqta uchun $m_i \le f(\xi_i) \le M_i$ va barcha $\Delta x_i > 0$ bo'lganda, $m_i\Delta x_i \le f(\xi_i)\Delta x_i \le$ $\le M_i\Delta x_i$ demak, $\sum_{i=1}^{n} m_i\Delta x_i \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} M_i\Delta x_i$ yoki $\underline{s_n} \le s_n \le \overline{s_n}$.

Bundan ko'rinadiki, yuzi s_n ga teng bo'lgan shakl ichki va tashqi chizilgan siniq chiziq orasida yotuvchi siniq chiziq bilan chegaralangan, degan ma'noni beradi. s_n yig'indining qiymati $[a;\ b]$ kesmani $[x_{i-1};\ x_i]$ kesmalarga ajratish usuliga hamda hosil qilingan kesmani ichida ξ_i nuqtalarni tanlab olishga bog'liq. Endi $\max[x_{i-1};\ x_i]$ bilan kesmalarni eng uzunini belgilaymiz va $\max[x_{i-1};\ x_i]$ nolga intiladigan holni qaraymiz. Har bir ajratish uchun ξ_i ning mos qiymatini tanlab, $s_n = \sum_{i=x}^{n} f(\xi_i)\Delta x_i$ integral yig'indini tuzamiz.

$n\to\infty$ intilganda $\max\Delta x_i\to 0$ bo'ladigan ketma-ketlikni qaraymiz va u biror limitga ega bo'lsin: $\lim\limits_{\max\Delta x_i\to 0} s_n = \lim\limits_{\max\Delta x_i\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = s.$

1-ta'rif. Agar $[a;\ b]$ kesma $\max\Delta x_i\to 0$ shartni qanoatlantiradigan har qanday bo'laklarga ajratilganda va $[x_{i-1};\ x_i]$ kesmada ξ_i ni istalgancha tanlab olganda $s_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ integral yig'indi birgina limitga intilsa, bu limit $[a;\ b]$ kesmada $f(x)$ funksiyaning *aniq integrali* deb ataladi va $\int_a^b f(x)\,dx$ bilan belgilanadi. Shunday qilib, ta'rifga ko'ra:

$$\lim_{\max\Delta x_i\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx,$$

a son integralning *quyi chegarasi*, b son esa integralning *yuqori chegarasi* deyiladi. $[a;\ b]$ — *integrallash kesmasi*, x esa *integrallash o'zgaruvchisi* deyiladi.

2-ta'rif. Agar $f(x)$ funksiya uchun yuqoridagi limit mavjud bo'lsa, u holda funksiya $[a;\ b]$ kesmada *integrallanuvchi* funksiya deyiladi.

Agar integral ostidagi $y = f(x)$ funksiyaning grafigini chizsak, $f(x) \geq 0$ bo'lgan holda $\int_a^b f(x)\,dx$ integralning son qiymati $y = f(x)$ egri chiziq, $x = a$, $x = b$ to'g'ri chiziqlar hamda Ox o'qi bilan chegaralangan egri chiziqli trapetsiya yuziga teng.

Teorema. (Teoremani isbotsiz keltiramiz.) *Agar f(x) funksiya [a; b] kesmada uzluksiz bo'lsa, u holda bu funksiya shu kesmada integrallanuvchidir. Uziluvchan funksiyalar orasida integrallanuvchi funksiyalar va integrallanmovchi funksiyalar ham bo'lishi mumkin.*

Eslatma. 1) aniq integral faqat $f(x)$ funksiyaning turiga va integralning chegarasiga bog'liq, ammo har qanday harf bilan belgilanishi mumkin bo'lgan integrallash o'zgaruvchisiga bog'liq emas:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \ldots = \int_a^b f(z)\,dz.$$

2) $\int_a^b f(x)\,dx$ aniq integral tushunchasini berishda $a < b$ deb faraz qildik. Agar $b < a$ bo'lsa, ta'rifga ko'ra:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \quad \text{ya'ni} \quad \int_6^0 x^3 dx = -\int_0^6 x^3 dx.$$

3) agar $a = b$ bo'lsa, ta'riflarga ko'ra, har qanday funksiya uchun tubandagi tenglik o'rinli bo'ladi:

$$\int_a^b f(x)\,dx = 0.$$

$y = f(x)$ funksiya $[a; b]$ kesmada aniqlangan va uzluksiz bo'lsin. U holda $\int_a^b f(x)\,dx$ mavjud va quyidagi xossalar o'rinli.

1-xossa. O'zgarmas ko'paytuvchini aniq integral belgisi tashqarisiga chiqarish mumkin, agar $C = \text{const}$ bo'lsa, u holda

$$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx.$$

Isbot.

$$\int_a^b Cf(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} Cf(\xi_i)\Delta x_i =$$

$$= C \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = C\int_a^b f(x)dx.$$

2-xossa. Bir necha funksiyalar algebraik yig'indisining aniq integrali qo'shiluvchilar aniq integrallarining algebraik yig'indisiga teng:

$$\int_a^b [f_1(x) + f_2(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)dx.$$

Isbot. Xossani ikkita qo'shiluvchi bo'lgan hol uchun isbotlaymiz.

$$\int [f_1(x) + f_2(x)]dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} [f_1(\xi_i) + f_2(\xi_i)]\Delta x_i =$$

$$= \lim_{\max \Delta x \to 0} \left[\sum_{i=1}^{n} f_1(\xi_i)\Delta x_i + \sum_{i=1}^{n} f_2(\xi_i)\Delta x_i\right] =$$

$$= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f_1(\xi_i)\Delta x_i + \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f_2(\xi_i)\Delta x_i =$$

$$= \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

Qo'shiluvchilar soni har qancha bo'lganda ham shunday isbot qilinadi.

3-xossa (bu xossa $a \ge b$ bo'lgandagina bajariladi). Agar $[a; b]$ $(a < b)$ kesmada $f(x)$ va $\varphi(x)$ funksiyalar $f(x) \le \varphi(x)$ shartni qanoatlantirsa, u holda $\int_a^b f(x)dx \le \int_a^b \varphi(x)dx$ o'rinli.

Isbot. Tubandagi ayirmani qaraymiz:

$$\int_a^b \varphi(x)dx - \int_a^b f(x)dx = \int_a^b [\varphi(x) - f(x)]\,dx =$$

$$= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} [\varphi(\xi_i) - f(\xi_i)]\Delta x_i,$$

bunda: $\varphi(\xi_i) - f(\xi_i) \ge 0$, $\Delta x_i \ge 0$, demak, butun yig'indi manfiy emas va uning limiti ham manfiy emas, ya'ni

$$\int_a^b [\varphi(x) - f(x)]\,dx \ge 0 \quad \text{yoki} \quad \int_a^b \varphi(x)dx - \int_a^b f(x)\,dx \ge 0.$$

Xossa isbot qilindi.

4-xossa. Agar M va m sonlar $f(x)$ funksiyaning $[a; b]$ kesmadagi eng katta va eng kichik qiymatlari bo'lib, $a \le b$ bo'lsa, u holda

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a) \text{ bo'ladi.}$$

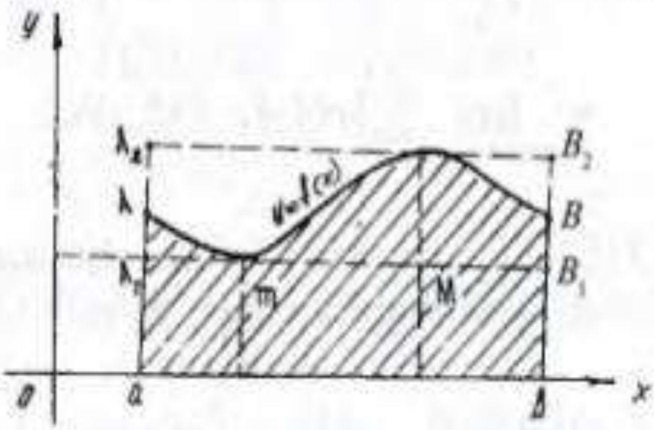
Isbot. Teoremaning shartiga ko'ra: $m \le f(x) \le M$;

3-xossaga ko'ra: $\int_a^b mdx \le \int_a^b f(x)\,dx \le \int_b^b Mdx$, bunda $\int_a^b mdx$, $\int_a^b Mdx$ ning qiymatlari mos ravishda $\int_a^b mdx = m(b-a)$ va $\int_a^b Mdx = M(b-a)$ ga teng.

Agar $f(x) \ge 0$ bo'lsa, u holda bu xossani geometrik usulda tasvirlasak, egri chiziqli $aABb$ trapetsiyaning yuzi, aA_1B_1b va aA_2B_2b to'g'ri to'rtburchaklar orasida yotadi (120-chizma).

5-xossa (o'rta qiymat haqida teorema). Agar $f(x)$ funksiya $[a; b]$ kesmada uzluksiz bo'lsa, u holda bu kesmada shunday bir c nuqta topiladiki, bu nuqta uchun $\int_a^b f(x)dx = (b-a)f(c)$ tenglik o'rinlidir.

Isbot. Aniqlik uchun $a < b$ bo'lgan holni qaraymiz. Agar m va M lar $f(x)$ ning $[a; b]$ kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda oldingi xossaga ko'ra $m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$, bundan $\frac{1}{b-a}\int_a^b f(x)dx = \mu$, bunda $m \le \mu \le M$; $f(x)$ uzluksiz funksiya bo'lgani uchun m va M orasidagi hamma oraliq qiymatlarni qabul qiladi.



Demak, biror c $(a \le c \le b)$ qiymatda $\mu = f(c)$ bo'ladi, ya'ni $\int_a^b f(x)dx = f(c)(b-a)$.

6-xossa. Agar quyidagi uchta integralning har biri mavjud bo'lsa, u holda har qanday uchta a, b, c son uchun

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik o'rinli bo'ladi.

Izoh. Isbot etilgan teoremadan xususiy holda har qanday uzluksiz funksiya boshlang'ich funksiyaga ega, degan natija kelib chiqadi.

Nyuton—Leybnits formulasi

2-teorema. *Agar $f(x)$ funksiya $[a; b]$ kesmada uzluksiz va $F(x)$ uzluksiz $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsa, u holda*

$$\int_a^b f(x)dx = F(b) - F(a)$$

formula o'rinlidir. Bu formula Nyuton—Leybnits formulasi deyiladi.

Isbot. $F(x)$ funksiya $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsin, 1-teoremaga muvofiq $\int_a^b f(t)dt$ funksiya ham $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladi. Ammo, berilgan funksiyaning har qanday 2 ta boshlang'ich funksiyasi bir-biridan o'zgarmas C^* qo'shiluvchi bilan farq qiladi:

$$\int_a^b f(t)dt = F(x) + C^*.$$

O'zgarmas C^* ni aniqlash uchun $x = a$ deb olamiz.

$\int_a^a f(t)\,dt = F(a) + C^*;\ 0 = F(a) + C^*;\ C^* = -F(a)$, demak, $\int_a^x f(t)\,dt = F(x) - F(a)$; $x = b$ deb olsak, Nyuton—Leybnits formulasi hosil bo'ladi:

$\int_a^b f(t)\,dt = F(b) - F(a)$; t ni x bilan almashtirsak, $\int_a^b f(x)\,dx = F(b) - F(a)$:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Integral ostidagi funksiyaning boshlang'ich funksiyasi ma'lum bo'lsa, u holda Nyuton—Leybnits formulasi aniq integralni hisoblash uchun juda qulaydir.

Aniq integralni hisoblashda ham, aniqmas integralni hisoblashdagidek, o'rniga qo'yish usuli yoki o'zgaruvchini almashtirish usulidan keng foydalaniladi.

Teorema. *$f(x)$ funksiya $[a; b]$ kesmada berilgan va uzluksiz bo'lsin. $\int_a^b f(x)dx$ integralni hisoblash talab qilinsin. $x = \varphi(t)$ o'zgaruvchini kiritamiz. $\varphi(t)$ funksiya quyidagi shartlarni qanoatlantirsin:*

1) $\varphi(t)$ funksiya $[\alpha; \beta]$ kesmada aniqlangan va uzluksiz;

2) $\varphi(\alpha) = a$; $\varphi(\beta) = b$;

3) $\varphi(t)$ funksiya $[\alpha; \beta]$ kesmada uzluksiz $\varphi'(t)$ hosilaga ega. U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt \qquad (1)$$

bo'ladi.

Isbot. Agar $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, quyidagi tenglikni yozish mumkin:

$$\int f(x)dx = F(x) + C. \qquad (2)$$

$$\int f[\varphi(t)]\varphi'(t)dt = F[\varphi(t)] + C. \qquad (3)$$

Keyingi tenglikning to'g'riligi uning ikki tomonini t bo'yicha differensiallash bilan tekshiriladi. Nyuton–Leybnits formulasiga ko'ra:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Bunga asosan:

$$(3) \Rightarrow \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt = F[\varphi(t)]\Big|_\alpha^\beta =$$

$$= F[\varphi(\beta)] - F[\varphi(\alpha)] = F(b) - F(a)$$

ekani kelib chiqadi. Keyingi ifodalarning o'ng tomonlari teng bo'lgani uchun chap tomonlari ham teng. Aniq integralni birinchi formula bilan hisoblagandan keyin eski o'zgaruvchiga o'tish zaruriyati yo'q.

M i s o l. $\int_0^r \sqrt{r^2 - x^2}\,dx$ integralni hisoblang.

Y e c h i s h. O'zgaruvchini almashtiramiz: $x = r\sin t$, $dx = r\cos t dt$, integrallashning yangi chegaralarini topamiz: $x = 0$ bo'lganda, $t = 0$; $x = r$ bo'lganda, $t = \frac{\pi}{2}$. Demak,

$$\int_0^r \sqrt{r^2 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t}\, r\cos t dt =$$

$$= r^2 \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = r^2 \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4}.$$

Aytaylik, $u = u(x)$ va $v = v(x)$ funksiyalar $[a; b]$ kesmada aniqlangan, uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. U holda $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$ bo'ladi.

Bu yerda: $u(x)v(x)$ funksiya $u'(x)v(x) + u(x)v'(x)$ funksiyaning boshlang'ich funksiyasi. Nyuton—Leybnits formulasiga asosan, bu ayniyatning ikkala tomonini a dan b gacha chegaralarda integrallaymiz: $\int_a^b (uv)'dx = \int_a^b u'v dx + \int_a^b uv' dx$, bunda $\int (uv)'dx = uv + C$

bo'lgani sababli, $\int_a^b (uv)'dx = uv\Big|_a^b$ o'rinli.

Demak, $uv\Big|_a^b = \int_a^b v du + \int_a^b u dv$ yoki $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du.$

Oxirgi tenglik aniq integralni *bo'laklab integrallash* formulasi deyiladi.

M i s o l. $\int_0^1 xe^{-x} dx$ integralni hisoblang.

Y e c h i s h. Belgilashlar kiritamiz: $u = x$; $dv = e^{-x}dx$, u holda $du = dx$; $v = -e^x$. Bo'laklab integrallash formulasiga ko'ra:

$$\int_0^1 xe^{-x} dx = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x}\Big|_0^1 = -2e^{-1} + 1 = \frac{e-2}{e}.$$

Mavzu: Xosmas integrallar. Chegaralari cheksiz xosmas integrallar.

Aniq integralga ta'rif berishda integrallash oralig'i *[a,b]* chekli va integral ostidagi funksiya shu oraliqda chegaralangan deb faraz qilgan edik. Ana shu shartlardan aqalli birortasi bajarilmasa aniq integralning keltirilgan ta'rifi ma'nosini yo'qotadi. Chunki integrallash oralig'i cheksiz bo'lganda uni uzunliklari chekli bo'lgan *n* ta qismga ajratib bo'lmaydi, integral ostidagi funksiya chegaralanmaganda integral yig'indi chekli limitga ega bo'lmaydi. Ammo aniq integral tushunchasini bu hollar uchun ham umumlashtirish mumkin. Umumlashtirish natijasida xosmas integrallar tushunchasiga kelamiz. Xosmas integrallar ikki turga-chegaralari cheksiz xosmas integrallar hamda chegaralanmagan funksyaning xosmas integraliga bo'linadi.

1-ta'rif. *f(x)* funksiya *[a,∞)* intervalda aniqlangan bo'lib, u istal-gan chekli *[a,R] (R>a)* kesmada integrallanuvchi, ya'ni $\int_a^R f(x)dx$ aniq integral mavjud bo'lsin. U holda

$$\lim_{R\to+\infty}\int_a^R f(x)dx \qquad (39.1)$$

chekli limit mavjud bo'lsa, u **birinchi tur** yoki **chegaralari cheksiz** xosmas integral deb ataladi, va

$$\int_a^{+\infty} f(x)dx \qquad (39.2)$$

kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra

Xosmas integrallar

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty}\int_a^t f(x)\,dx.$$

1. Xosmas integralni hisoblang.

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty}\int_0^b e^{-x}dx = \lim_{b\to+\infty}(e^{-x})\Big|_0^b = -\lim_{b\to+\infty}(e^{-x})\Big|_0^b =$$

$$= -\lim_{b\to+\infty} e^{-b} + \lim_{b\to+\infty} e^0 = -\lim_{b\to+\infty}\frac{1}{e^b} + 1 = 0 + 1 = 1.$$

2. Xosmas integralni hisoblang.

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2} =$$

$$= \lim_{a\to-\infty} arctg \Big|_a^0 + \lim_{b\to+\infty} arctgx \Big|_0^b = \lim_{a\to-\infty} (arctg - arctga) +$$

$$+ \lim_{b\to+\infty} (arctgb - arctg0) = - \lim_{a\to-\infty} arctga + \lim_{b\to+\infty} arctgb =$$

$$= -arctg(-\infty) + arctg(+\infty) = -\left(\frac{\pi}{2}\right) + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

3. Xosmas integralni hisoblang.

$$\int_{0}^{+\infty} \frac{dx}{x^2+4}$$

Xosmas integralning yaqinlashuvchiligini ko‘rsating va qiymatini toping.

Yechish: $f(x) = \frac{1}{x^2+4}$ funksiya har bir chekli $[0, A]$ $(A > 0)$ oraliqda integrallanuvchi.

$$\int_{0}^{+\infty} \frac{dx}{x^2+4} = \lim_{A\to+\infty} \left(\frac{1}{2} arctg \frac{A}{2} - \frac{1}{2} arctg \frac{0}{2}\right) = \lim_{A\to+\infty} \left(\frac{1}{2} arctg \frac{A}{2}\right) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Demak, berilgan xosmas integral yaqinlashuvchi va $\frac{\pi}{4}$ qiymatga ega.

4. Xosmas integralni hisoblang.

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+9}$$

Xosmas integralning yaqinlashuvchiligini ko‘rsating va qiymatini toping.

Yechish: Xosmas integralning ta’rifiga ko‘ra,

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+9} = \int_{-\infty}^{0} \frac{dx}{x^2+4x+9} + \int_{0}^{+\infty} \frac{dx}{x^2+4x+9} =$$

$$= \lim_{A\to-\infty} \int_{A}^{0} \frac{dx}{x^2+4x+9} + \lim_{A'\to-\infty} \int_{0}^{A'} \frac{dx}{x^2+4x+9} =$$

$$= \lim_{A\to-\infty} \left(\frac{1}{\sqrt{5}} arctg \frac{x+2}{\sqrt{5}} \Big|_A^0 \right) + \lim_{A'\to+\infty} \left(\frac{1}{\sqrt{5}} arctg \frac{x+2}{\sqrt{5}} \Big|_0^{A'} \right) =$$

$$= \frac{1}{\sqrt{5}} arctg \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} \lim_{A\to-\infty} arctg \frac{A+2}{\sqrt{5}} + \frac{1}{\sqrt{5}} \lim_{A'\to+\infty} arctg \frac{A'+2}{\sqrt{5}} - \frac{1}{5} arctg \frac{2}{\sqrt{5}} =$$

$$= -\frac{1}{\sqrt{5}} \cdot \left(-\frac{\pi}{2}\right) + \frac{1}{\sqrt{5}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{5}}.$$

Demak, berilgan xosmas integral yaqinlashuvchi va $\int_{0}^{+\infty} \frac{dx}{x^2+4x+9} = \frac{\pi}{\sqrt{5}}$.

5. Xosmas integralni hisoblang.

$$\int_{0}^{+\infty} cos4x\, dx$$

Xosmas integralning yaqinlashuvchiligini ko‘rsating.

Yechish: Xosmas integralning ta’rifiga ko‘ra,

$$\int_{0}^{+\infty} \cos 4x dx = \lim_{A\to+\infty}\int_{0}^{A} \cos 4x dx = \lim_{A\to+\infty}\left(\frac{1}{4}\sin 4x\Big|_{0}^{A}\right) = \lim_{A\to+\infty}\frac{1}{4}\sin 4A,$$

Bu limit mavjud bo‘lmagani uchun berilgan xosmas integral uzoqlashuvchi bo‘ladi.

6-misol. $\int_{1}^{\infty}\frac{\cos^2 3x}{\sqrt[3]{x^5+1}}dx$ integralni yaqinlashuvchilikka tekshiring.

Yechish. $[1, +\infty]$ da quyidagi tengsizlik o’rinli

$$0 \le \frac{\cos^2 3x}{\sqrt[3]{x^5+1}} < \frac{1}{\sqrt[3]{x^5}},$$

va

$$\int_{1}^{\infty}\frac{dx}{\sqrt[3]{x^5}} = \frac{x^{-\frac{5}{3}+1}}{-\frac{5}{3}+1}\Bigg|_{1}^{\infty} = -\frac{3}{2x^{\frac{2}{3}}}\Bigg|_{1}^{\infty} = \frac{3}{2},$$

u holda 1-teoremaga asosan, berilgan integral yaqinlashuvchi bo’ladi.

7-misol. $\int_{1}^{+\infty}\frac{\cos x}{x^2}dx$ xosmas integralning absolyut yaqinlashuvchiligini isbotlang.

Yechish. Ushbu $\int_{1}^{+\infty}\left|\frac{\cos x}{x^2}\right|dx$ integralni qaraymiz. Ravshanki,

$\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$, $\forall x \in [1, +\infty)$ va $\int_{1}^{+\infty}\frac{1}{x^2}dx = \left(-\frac{1}{x}\right)\Big|_{1}^{+\infty} = 1$ ekanligidan, 1-taqqoslash teoremasiga ko’ra, $\int_{1}^{+\infty}\left|\frac{\cos x}{x^2}\right|dx$ integral yaqinlashuvchi bo’ladi. Demak, berilgan integral absolyut yaqinlashuvchi ekan.

8. Xosmas integralni hisoblang.

$$\int_2^{+\infty}\left(\frac{1}{x^2-1}+\frac{2}{(x+1)^2}\right)dx$$

Yechish. $f(x)=\frac{1}{x^2-1}$ va $g(x)=\frac{2}{(x+1)^2}$ funksiyalar uchun quyidagi funksiyalar boshlang'ich funksiya bo'ladi.

$$F(x)=\frac{1}{2}\ln\frac{x-1}{x+1} \quad \text{va} \quad C(x)=-\frac{2}{(x+1)}$$

Nyuton-Leybnis formulasiga ko'ra topamiz:

$$\int_2^{+\infty}\frac{dx}{x^2-1}=-\frac{1}{2}\ln\frac{x-1}{x+1}\bigg|_2^{+\infty}=-\frac{1}{2}\ln\frac{1}{3}=\frac{1}{2}\ln 3,$$

$$\int_2^{+\infty}\frac{2dx}{(x+1)^2}=-\frac{2}{x+1}\bigg|_2^{+\infty}=\frac{2}{3}$$

Demak, 1-xossaga asosan, berilgan integral $\frac{2}{3}+\frac{1}{2}ln3$ ga teng bo'ladi.

9-misol. $\int_0^1\frac{dx}{x^\alpha}$, $\alpha\in R$ integralni yaqinlashishga tekshiring.

Yechish. $\alpha\neq 1$ bo'lsin, u holda

$$\int_0^1\frac{dx}{x^\alpha}=\lim_{\xi\to+0}\int_\xi^1\frac{dx}{x^\alpha}=\lim_{\xi\to+0}\frac{x^{-\alpha+1}}{1-\alpha}\bigg|_\xi^1=\lim_{\xi\to+0}\frac{1-\xi^{1-\alpha}}{1-\alpha}=\begin{cases}\frac{1}{1-\alpha}, & \text{агар } \alpha<1 \text{ булса},\\ +\infty, & \text{агар } \alpha>1 \text{ булса}.\end{cases}$$

Endi $\alpha=1$ bo'lsin, u holda

$$\int_0^1\frac{dx}{x}=\lim_{\xi\to+0}\int_\xi^1\frac{dx}{x}=\lim_{\xi\to+0}\ln|x|\,\Big|_\xi^1=-\lim_{\xi\to+0}\ln\xi=+\infty.$$

Ravshanki, $\int_0^1\frac{dx}{x^\alpha}$ integral $\alpha<1$ bo'lsa yaqinlashuvchi va $\alpha\geq 1$ bo'lsa uzoqlashuvchi bo'ladi.

10-misol. $\int_1^2\frac{dx}{x\ln x}$ xosmas integralning uzoqlashuvchi ekanligini ko'rsating.

Yechish. $\int_1^2\frac{dx}{x\ln x}=\lim_{\xi\to 1+0}\int_\xi^2\frac{dx}{x\ln x}=\lim_{\xi\to 1+0}(\ln(\ln x))\Big|_\xi^2=\lim_{\xi\to 1+0}(\ln(\ln 2)-\ln(\ln\xi))=+\infty.$

Demak, berilgan integral uzoqlashuvchi ekan.

11. Xosmas integralni hisoblang.

$$\int_1^{+\infty} \frac{arcctgx}{x^2}dx$$

Yechish: Berilgan xosmas integralga bo'laklab integrallash formulasini qo'llasak, $u = arcctgx, dv = \frac{1}{x^2}dx$ deyilsa, u holda $du = \frac{dx}{1+x^2}, v = -\frac{1}{x^2}$ bo'lib, quyidagiga ega bo'lamiz:

$$\int_1^{+\infty} \frac{arcctgx}{x^2}dx = -\frac{arcctgx}{x}\Bigg|_1^{+\infty} - \int_1^{\infty}\left(-\frac{1}{x}\right)\left(-\frac{1}{1+x^2}dx\right) =$$

$$= \frac{\pi}{4} - \int_1^{\infty} \frac{dx}{x(x^2+1)} = \frac{\pi}{4} - \int_1^{\infty}\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \frac{\pi}{4} - \left(\ln x - \frac{1}{2}\ln(x^2+1)\right)\Bigg|_1^{+\infty} =$$

$$= \frac{\pi}{4} - \ln\frac{x}{\sqrt{x^2+1}}\Bigg|_1^{+\infty} = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

Quyidagi xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanligini ko'rsating:

1. $\int_2^{+\infty} \frac{dx}{x^2}$;
2. $\int_1^{+\infty} \frac{dx}{2\sqrt{x}}$;
3. $\int_2^{+\infty} \frac{dx}{x^2+x-2}$;
4. $\int_1^{+\infty} \frac{dx}{\sqrt{4+x^2}}$;
5. $\int_0^{+\infty} e^{-ax}\cos bx dx, \quad (a>0)$;
6. $\int_{-\infty}^{0} xe^x dx$;
7. $\int_1^{+\infty} e^{-3x}dx$;
8. $\int_{-\infty}^{0} \frac{x+1}{x^2+1}dx$;
9. $\int_{-\infty}^{0} \frac{dx}{x+1}$;
10. $\int_{-\infty}^{\infty} \frac{dx}{2x^2-5x+7}$;
11. $\int_e^{+\infty} \frac{dx}{x\ln x}$;
12. $\int_3^{+\infty} \frac{2x+5}{x^2+3x-10}dx$;
13. $\int_2^{+\infty} x\cdot 2^{-x}dx$;
14. $\int_2^{+\infty} \frac{dx}{x^2-1}$;

15. $\int_{1}^{+\infty} \frac{arctgx}{1+x^2} dx$;

16. $\int_{e}^{+\infty} \frac{dx}{x \ln^2 x}$;

17. $\int_{0}^{+\infty} \frac{dx}{x\sqrt{x}}$;

18. $\int_{1}^{+\infty} x \cdot \cos x dx$;

19. $\int_{0}^{+\infty} \sin 2x dx$;

20. $\int_{1}^{+\infty} \frac{dx}{x\sqrt{x^2+x+1}}$;

21. $\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2} \sin\frac{1}{x} dx$;

22. $\int_{-\infty}^{0} e^{5x} dx$;

23. $\int_{0}^{+\infty} e^{-ax} \sin bx dx, \quad (a > 0)$;

24. $\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$.

Mavzu: Xosmas integrallar.

Aniq integralga ta'rif berishda integrallash oralig'i *[a,b]* chekli va integral ostidagi funksiya shu oraliqda chegaralangan deb faraz qilgan edik. Ana shu shartlardan aqalli birortasi bajarilmasa aniq integralning keltirilgan ta'rifi ma'nosini yo'qotadi. Chunki integrallash oralig'i cheksiz bo'lganda uni uzunliklari chekli bo'lgan *n* ta qismga ajratib bo'lmaydi, integral ostidagi funksiya chegaralanmaganda integral yig'indi chekli limitga ega bo'lmaydi. Ammo aniq integral tushunchasini bu hollar uchun ham umumlashtirish mumkin. Umumlashtirish natijasida xosmas integrallar tushunchasiga kelamiz. Xosmas integrallar ikki turga- chegaralari cheksiz xosmas integrallar hamda chegaralanmagan funksyaning xosmas integraliga bo'linadi.

1-ta'rif. $f(x)$ funksiya $[a,\infty)$ intervalda aniqlangan bo'lib, u istalgan chekli $[a,R]$ $(R>a)$ kesmada integrallanuvchi, ya'ni $\int_a^R f(x)dx$ aniq integral mavjud bo'lsin. U holda

$$\lim_{R\to+\infty}\int_a^R f(x)dx \qquad (39.1)$$

chekli limit mavjud bo'lsa, u **birinchi tur** yoki **chegaralari cheksiz** xosmas integral deb ataladi, va

$$\int_a^{+\infty} f(x)dx \qquad (39.2)$$

kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra

$$\int_a^{+\infty} f(x)dx = \lim_{R\to+\infty}\int_a^R f(x)dx .$$

Bu holda (39.2.) xosmas integral **mavjud** yoki **yaqinlashadi** deyiladi. Agar (39.1) limit mavjud bo'lmasa, u holda (39.2) xosmas integral **mavjud emas** yoki **uzoqlashadi** deyiladi.

$(-\infty,b]$ intervalda chegaralangan $f(x)$ funksiyaning xosmas integrali ham (39.2) kabi aniqlanadi:

$$\int_{-\infty}^{b} f(x)dx = \lim_{R\to-\infty}\int_R^b f(x)dx . \qquad (39.3)$$

Bunda $f(x)$ funksiya istalgan $[R,b]$ $(R<b)$ kesmada integral-lanuvchi.

Shuningdek,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx \qquad (39.4)$$

tenglik yordamida *f(x)* funksiyaning $(-\infty, +\infty)$ bo'yicha xosmas integrali aniqlanadi. Bunda *c* ixtiyoriy o'zgarmas son. (39.4) tenglikning o'ng tomonidagi har ikkala xosmas integrallar yaqinlashgada uning chap tomonidagi xosmas interal ham yaqinlashadi.

Endi birinchi tur xosmas integralni hisoblashga misollar keltiramiz.

1- misol. $\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^2} = \lim_{R\to\infty} \operatorname{arctg} x \Big|_0^R = \lim_{R\to\infty} \operatorname{arctg} R = \frac{\pi}{2}$,

ya'ni berilgan xosmas integral yaqinlashadi.

2- misol.

$$\int_0^{+\infty} \sin x dx = \lim_{R\to+\infty} \int_0^R \sin x dx = \lim_{R\to+\infty} (-\cos x) \Big|_0^R = \lim_{R\to+\infty} (-\cos R + 1) =$$

$$= 1 - \lim_{R\to+\infty} \cos R,$$

ammo $\cos R$ funksiya $R \to +\infty$ da limitga ega bo'lmaganligi uchun integral uzoqlashadi.

3-misol. $\int_1^{+\infty} \frac{dx}{x^\alpha}$, α -biror son.

Shu birinchi tur xosmas integralni α ning qanday qiymatlarida yaqinlashishi va qanday qiymatlarida uzoqlashishini aniqlaymiz.

a) $\alpha \neq 1$ bo'lsin, u holda istalgan $R>0$ uchun

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{R\to+\infty} \int_1^R \frac{dx}{x^\alpha} = \lim_{R\to+\infty} \int_1^R x^{-\alpha} dx = \lim_{R\to+\infty} \frac{x^{-\alpha+1}}{-\alpha+1} \Big|_1^R =$$

$$= \lim_{R\to+\infty} \frac{R^{1-\alpha} - 1}{1-\alpha} = \begin{cases} \frac{1}{\alpha - 1}, \text{agar } \alpha > 1, \text{bo' lsa}, \\ +\infty, \text{agar } \alpha < \text{bo' lsa}. \end{cases}$$

b) $\alpha = 1$ bo'lsin, u holda istalgan $R>0$ son uchun

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{R\to+\infty} \int_1^R \frac{dx}{x} = \lim_{R\to+\infty} \ln x \Big|_1^R = \lim_{R\to+\infty} \ln R = +\infty.$$

Shunday qilib, berilgan integral $\alpha > 1$ bo'lganda yaqinlashadi, $\alpha \leq 1$ bo'lganda esa uzoqlashadi.

2-ta'rif. *f(x)* funksiya *[a,b)* oraliqda aniqlangan bo'lsin. Agar *f(x)* funksiya $x=b$ nuqtaning biror atrofida chegaralanmagan bo'lib, u *[a,b)* ga tegishli har qanday $[a, b-\varepsilon]$ kesmada chegaralangan bo'lsa $x=b$ nuqta *f(x)* ning **maxsus** nuqtasi deyiladi. $x=b$ nuqta *f(x)*ning maxsus nuqtasi bo'lib *f(x)* funksiya istalgan $[a, b-\varepsilon]$ $(\varepsilon > 0, b-\varepsilon > a)$ kesmada integrallashuvchi, ya'ni $\int_a^{b-\varepsilon} f(x)dx$ aniq integral mavjud bo'lsin. U holda

$$\lim_{\varepsilon\to+0} \int_a^{b-\varepsilon} f(x)dx \qquad (39.5)$$

chekli limit mavjud bo'lsa, uni **ikkinchi tur** yoki **chegaralanmagan funksiyaning** xos integrali deb ataladi va

$$\int_a^b f(x)dx \qquad (39.6)$$

ko'rinishda belgilanadi. Bu holda (39.6) integral mavjud yoki **yaqinlashadi** deb aytiladi. (39.5) limit mavjud bo'lmasa, u holda (39.6) integral mavjud emas yoki **uzoqlashadi** deb aytiladi.

Shuningdek $x=a$ nuqta $f(x)$ funksiyaning maxsus nuqtasi (a nuqtaning yaqin atrofida $f(x)$ chegaralanmagan va istalgan $[a+\varepsilon,b]$ $(\varepsilon>0, a+\varepsilon<b)$ kesmada chegaralangan) bo'lganda xosmas integral

$$\int_a^b f(x)dx = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b f(x)dx$$

kabi aniqlanadi.

Agar $f(x)$ funksiya $[a,b]$ kesmaning biror ichki c nuqtasining qandaydir atrofida chegaralanmaganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik yordamida aniqlanadi. Bu tenglikning o'ng tomonidagi har ikkala integral yaqinlashganda uning chap tomonidagi xosmas integral ham yaqinlashadi.

Shuningdek, a va b nuqtalar $f(x)$ funksiyaning maxsus nuqtalari bo'lganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (39.7)$$

tenglik yordamida aniqlanadi, bunda c (a,b) intervalining ixtiyoriy nuqtasi.

Izoh. Uzluksiz funksiyaning ikkinchi tur uzilish nuqtasi uning maxsus nuqtasi bo'ladi.

4- misol. $\int_a^b \frac{dx}{(x-a)^\alpha}$ integral tekshirilsin, bunda $a>0$ - biror son.

Yechish. Integral ostidagi $\frac{1}{(x-a)^{\alpha}}$ funksiya uchun $x=a$ maxsus nuqta bo'ladi.

a) $\alpha \neq 1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{(x-a)^{\alpha}} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{(x-a)^{\alpha}} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b (x-a)^{-\alpha}dx = \lim_{\varepsilon\to+0}\frac{(x-a)^{-\alpha+1}}{-\alpha+1}\Big|_{a+\varepsilon}^b =$$

$$= \begin{cases} \dfrac{(b-a)^{1-\alpha}}{1-\alpha}, \text{agar } \alpha<1 \quad \text{bo'lsa}, \\ +\infty, \text{agar } \alpha>1 \quad \text{bo'lsa}. \end{cases}$$

b) $\alpha=1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\ln|x-a|\Big|_{a+\varepsilon}^b = \lim_{\varepsilon\to+0}\left[\ln(b-a)-\ln\varepsilon\right] = +\infty.$$

Shunday qilib berilgan integral $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

Shuningdek $\int_a^b \frac{dx}{(b-x)^{\alpha}}$ integral ham $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

5- misol. $\int_2^5 \frac{dx}{\sqrt[3]{x-3}}$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x-3}}$ funksiya *[2,5]* kesmaning ichidagi $x=3$ nuqtada ikkinchi tur uzilishga ega, ya'ni bu nuqta funksyaining maxsus nuqtasi.

Ta'rifga binoan:

$$\int_2^5 \frac{dx}{\sqrt[3]{x-3}} = \int_2^3 \frac{dx}{\sqrt[3]{x-3}} + \int_3^5 \frac{dx}{\sqrt[3]{x-3}} = \lim_{\varepsilon_1\to+0}\int_2^{3-\varepsilon_1}\frac{dx}{(x-3)^{\frac{1}{3}}} + \lim_{\varepsilon_2\to+0}\int_{3+\varepsilon_2}^5 \frac{dx}{(x-3)^{\frac{1}{3}}} =$$

$$= \lim_{\varepsilon_1\to+0}\frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\Big|_2^{3-\varepsilon_1} + \lim_{\varepsilon_2\to+0}\frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\Big|_{3+\varepsilon_2}^5 = \lim_{\varepsilon_1\to+0}\frac{3}{2}\sqrt[3]{(x-3)^2}\Big|_2^{3-\varepsilon_1} +$$

$$+ \lim_{\varepsilon_2\to+0}\frac{3}{2}\sqrt[3]{(x-3)^2}\Big|_{3+\varepsilon_2}^5 = \frac{3}{2}\lim_{\varepsilon_1\to+0}\left[\sqrt[3]{\varepsilon_1^2}-1\right] + \frac{3}{2}\lim_{\varepsilon_2\to+0}\left[\sqrt[3]{2^2}-\sqrt[3]{\varepsilon_2^2}\right] = -\frac{3}{2}+\frac{3}{2}\sqrt[3]{4} = \frac{3}{2}\left(\sqrt[3]{4}-1\right).$$

Demak, berilgan integral yaqinlashadi.

6-misol. $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt{x(1-x)}}$ funksiya uchun $a=0,\ b=1$ nuqtalar maxsus nuqtalar bo'ladi. Xosmas itntegralning ta'rifidan foydalanib topamiz:

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\varepsilon_1\to+0}\int_{\varepsilon_1}^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \lim_{\varepsilon_2\to+0}\int_{\frac{1}{2}}^{1-\varepsilon_2} \frac{dx}{\sqrt{x(1-x)}} =$$

$$= \lim_{\varepsilon_1\to+0}\int_{\varepsilon_1}^{\frac{1}{2}} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} + \lim_{\varepsilon_2\to+0}\int_{\frac{1}{2}}^{1-\varepsilon_2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \lim_{\varepsilon_1\to+0}\arcsin\frac{x-\frac{1}{2}}{\frac{1}{2}}\Big|_{\varepsilon_1}^{\frac{1}{2}} +$$

$$+ \lim_{\varepsilon_2\to+0}\arcsin\frac{x-\frac{1}{2}}{\frac{1}{2}}\Big|_{\frac{1}{2}}^{1-\varepsilon_2} = \lim_{\varepsilon_1\to+0}\left[\arcsin 0-\arcsin(2\varepsilon_1-1)\right] -$$

$$-\lim_{\varepsilon_2\to+0}\left[\arcsin(2\varepsilon_2-1)-\arcsin 0\right]=\arcsin 1+\arcsin 1=2\cdot\frac{\pi}{2}=\pi.$$

Demak integral yaqinlashuvchi va

$$\int_0^1\frac{dx}{\sqrt{x(1-x)}}=\pi.$$

7- misol. $\int_{-1}^{1}\frac{dx}{x^2}$ integral hisoblansin.

Yechish. Birinchi qarashda berilgan integral juda oson hisoblanadi, ya'ni

$$\int_{-1}^{1}\frac{dx}{x^2}=-\frac{1}{x}\Big|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2.$$

Ammo olingan natija noto'g'ri. Bu noto'g'ri natijaga e'tiborsizligimiz oqibatida, ya'ni integral ostidagi $\frac{1}{x^2}$ funksiya *[-1,1]* kesmada *x=0* maxsus nuqtaga ega ekanligini hisobga olmaganligimiz sababli keldik. Biz berilgan integralni oddiy aniq integral deb emas, balki xosmas integral deb qarashimiz lozim. Uni ta'rifdan foydalanib hisoblaymiz:

$$\int_{-1}^{1}\frac{dx}{x^2}=\int_{-1}^{0}\frac{dx}{x^2}+\int_{0}^{1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}+\lim_{\varepsilon_2\to+0}\int_{\varepsilon_2}^{1}\frac{dx}{x^2}. \qquad (39.8)$$

Limitlardan birini hisoblaymiz:

$$\lim_{\varepsilon_1\to+0}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\left(-\frac{1}{x}\right)\Big|_{-1}^{\varepsilon}=\lim_{\varepsilon\to+0}\left(+\frac{1}{\varepsilon}-\frac{1}{-1}\right)=\infty.$$

(39.8) tenglikning o'ng tomonidagi xosmas integrallardan biri uzoqlashganligi uchun ta'rifga binoan uning chap tomonidagi xosmas integral ham uzoqlashadi.

39.4. Xosmas integrallarning yaqinlashish alomatlari

Ko'p hollarda xosmas integralning qiymatini topish talab etilmasdan uning yaqinlashuvchi yoki uzoqlashuvchi ekanini bilishning o'zi kifoya qiladi. Bunday hollarda **taqqoslash teoremalari** deb ataluvchi quyidagi teoremalardan foydalanish mumkin.

39.1-teorema. Agar $f(x)$ va $\varphi(x)$ funksiyalar $[a,+\infty)$ oraliqda uzluksiz bo'lib, $0\le f(x)\le\varphi(x)$ shartni qanoatlantirsa, u holda

Ko'p hollarda xosmas integralning qiymatini topish talab etilmasdan uning yaqinlashuvchi yoki uzoqlashuvchi ekanini bilishning o'zi kifoya qiladi. Bunday hollarda **taqqoslash teoremalari** deb ataluvchi quyidagi teoremalardan foydalanish mumkin.

39.1-teorema. Agar $f(x)$ va $\varphi(x)$ funksiyalar $[a, +\infty)$ oraliqda uzluksiz bo'lib, $0 \le f(x) \le \varphi(x)$ shartni qanoatlantirsa, u holda

a) $\int_a^{+\infty} \varphi(x)dx$ xosmas integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

b) $\int_a^{+\infty} f(x)dx$ integrla uzoqlashganda $\int_a^{+\infty} \varphi(x)dx$ integral ham uzoqlashadi.

Bu teorema faqatgina nomanfiy funksiyalarga tegishli bo'lib undan ishorasini saqlamaydigan funksiyalarning xosmas integrallarini tekshirishda foydalanib bo'lmaydi. Bunday holda quyidagi teoremadan foydalanish mumkin.

39.2-teorema. $\int_a^{+\infty} |f(x)|dx$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

Bunda oxirgi integral **absolyut yaqinlashuvchi** deyiladi.

$\int_a^{+\infty} f(x)dx$ yaqinlashuvchi $\int_a^{+\infty} |f(x)|dx$ integral uzoqlashuvchi bo'lganda $\int_a^{+\infty} f(x)dx$ integral **shartli yaqinlashuvchi** deyiladi.

8- misol. $\int_1^{+\infty} \frac{dx}{x^3(1+e^x)}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{x^3(1+e^x)}$ funksiyani $\frac{1}{x^3}$ funksiya bilan taqqoslaymiz. Barcha $x \ge 1$ uchun $\frac{1}{x^3(1+e^x)} \le \frac{1}{x^3}$ bo'lib, $\int_1^{+\infty} \frac{dx}{x^3}$ yaqinlashganligi (3- misolga qarang) uchun 39.1. teoremaning *a)* bandiga binoan berilgan integral ham yaqinlashadi.

9- misol. $\int_{1}^{+\infty}\frac{\sqrt{x}}{1+x}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{\sqrt{x}}{1+x}$ funksiyani $\frac{1}{2\sqrt{x}}$ funksiya bilan taqqoslab barcha $x \geq 1$ uchun

$$\frac{1}{2\sqrt{x}}=\frac{\sqrt{x}}{2x}=\frac{\sqrt{x}}{x+x}\leq\frac{\sqrt{x}}{1+x}$$

ga ega bo'lamiz. $\alpha=\frac{1}{2}<1$ bo'lgani uchun $\int_{1}^{+\infty}\frac{dx}{2\sqrt{x}}$ integral uzoqlashadi (3-misolga qarang).

39.1. teoremaning *b)* qismiga ko'ra berilgan integral ham uzoqlashadi.

10-misol. $\int_{1}^{+\infty}\frac{\sin x}{x^4}dx$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\sin x}{x^4}$ funksiya $[1,+\infty)$ da ishorasini saqlamaydi. Shuning uchun $\int_{1}^{+\infty}\left|\frac{\sin x}{x^4}\right|dx$ integralni qaraymiz. $[1,+\infty)$ da

$$0\leq\left|\frac{\sin x}{x^4}\right|=\frac{|\sin x|}{x^4}\leq\frac{1}{x^4}$$

bajarilib $\int_{1}^{+\infty}\frac{dx}{x^4}$ (3-misolga qarang) yaqinlashganligi uchun 39.1-teoremaning *a)* bandiga ko'ra $\int_{1}^{+\infty}\left|\frac{\sin x}{x^4}\right|dx$ yaqinlashadi.

39.2-teoremaga ko'ra berilgan integral ham yaqinlashadi. U absolyut yaqinlashadi.

11-misol. $\int_{0}^{\infty}\frac{\sin x}{x}dx$

Dirixle integralining shartli yaqinlashuvchiligi ko'rsatilsin.

Yechish. Integralni ikkita integrallarning yig'indisi ko'rinishda tasvirlaymiz:

$$\int_0^{\infty}\frac{\sin x}{x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx+\int_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx.$$

$\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ bo'lgani uchun birinchi integral xos ma'noda ya'ni u aniq integral sifatida mavjud, chunki $\frac{\sin x}{x}$ funksiya $(0;\frac{\pi}{2}]$ oraliqda uzluksiz bo'lib $x=0$ nuqta uning yo'qotilishi mumkin bo'lgan uzilish nuqtasi.

Ikkinchi integralni bo'laklab integrallaymiz:

$$\int_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx=\lim_{A\to\infty}\int_{\frac{\pi}{2}}^{A}\frac{\sin x}{x}dx\left|\begin{array}{l}\frac{1}{x}=u,\ du=-\frac{1}{x^2}dx\\ \sin xdx=dv,\ v=-\cos x\end{array}\right|=$$

$$=\lim_{A\to\infty}\left[-\frac{\cos x}{x}\Bigg|_{\frac{\pi}{2}}^{A}-\int_{\frac{\pi}{2}}^{A}\frac{\cos x}{x^2}dx\right]=-\lim_{A\to\infty}\frac{\cos A}{A}-\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx=-\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx.$$

Ixtiyoriy x uchun $\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$ bajarilib $\int_{\frac{\pi}{2}}^{\infty}\frac{dx}{x^2}$ $(p=2>1)$ integral yaqinlashganligi sababli 39.1-teoremaning *a)* bandiga ko'ra $\int_{\frac{\pi}{2}}^{\infty}\frac{|\cos x|}{x^2}dx$ integral ham yaqinlashadi, demak $\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx$ absolyut yaqinlashadi.

Demak $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx$ xosmas integral ham absolyut yaqinlashadi.

Bu integral uchun yuritilgan mulohazalarni takrorlab $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x}dx$ integralni ham yaqinlashishini ko'rsatish mumkin.

Shunday qilib $\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin x}{x}dx$ aniq integral bilan yaqinlashuvchi $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx$ xosmas integralning yig'indisidan iborat $\int\limits_{0}^{\infty}\frac{\sin x}{x}dx$ integral ham yaqinlashadi.

Endi $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$ integralning uzoqlashuvchiligini ko'rsatamiz.

Barcha $x \geq \frac{\pi}{2}$ uchun

$$\frac{|\sin x|}{x} \geq \frac{\sin^2 x}{x} = \frac{1-\cos 2x}{2x}$$

bajarilib

$$\int\limits_{\frac{\pi}{2}}^{\infty}\frac{1-\cos 2x}{2x}dx = \frac{1}{2}\lim_{A\to+\infty}\int\limits_{\frac{\pi}{2}}^{A}\frac{dx}{x} - \frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{x}dx = \frac{1}{2}\lim_{A\to+\infty}\ln x\Big|_{\frac{\pi}{2}}^{A} - \frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{x}dx =$$

$$= \frac{1}{2}\lim_{A\to+\infty}\ln A - -\frac{1}{2}\ln\frac{\pi}{2} - \int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{2x}dx = \infty$$

xosmas integral uzoqlashganligi uchun 39.1-teoremaning *b)* bandiga ko'ra $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$ xosmas integral ham uzoqlashadi, bu yerda $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{2x}dx$ integralning yaqilashuvchanligi hisobga olinadi.

$$\int\limits_{0}^{\infty}\frac{|\sin x|}{x}dx = \int\limits_{0}^{\frac{\pi}{2}}\frac{|\sin x|}{x}dx + \int\limits_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$$

yig'indining birinchi integrali aniq integral bo'lib ikkinchi integrali uzoqlashuvchi xosmas integal bo'lganligi sababli $\int\limits_{0}^{\infty}\frac{|\sin x|}{x}dx$ integral ham uzoqlashadi.

Demak, $\int\limits_{0}^{\infty}\frac{\sin x}{x}dx$ xosmas integral shartli yaqinlashuvchi.

12-misol. $\int_0^{\infty}\sin x^2dx$, $\int_0^{\infty}\cos x^2dx$ xosmas integralning yaqinlashuvchiligi isbotlansin.

Yechish. $\int_0^{\infty}\sin x^2dx$ integralning yaqinlashuvchiligini ko'rsatamiz.

$x=\sqrt{t}$, $dx=\frac{dt}{2\sqrt{t}}$ almashtirish olib berilgan integralni ikkita integrallarning yig'indisi ko'rinishda tasvirlaymiz:

$$\int_0^{\infty}\sin x^2dx=\frac{1}{2}\int_0^{\infty}\frac{\sin t}{\sqrt{t}}dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin t}{\sqrt{t}}dt+\frac{1}{2}\int_{\frac{\pi}{2}}^{\infty}\frac{\sin t}{\sqrt{t}}dt,$$

$\lim\limits_{t\to0}\frac{\sin t}{\sqrt{t}}=\lim\limits_{t\to0}\frac{\sin t}{t}\cdot\sqrt{t}=1\cdot0=0$ bo'lganligi sababli xuddi 11-misoldagi kabi birinchi integral aniq integral sifatida mavjud. Ikkinchi integralni bo'laklab integrallaymiz:

$$\int_{\frac{\pi}{2}}^{\infty}\frac{\sin t}{\sqrt{t}}dt\left|\begin{array}{l}\frac{1}{\sqrt{t}}=u,\ \sin tdt=dv\\ du=(t^{-\frac{1}{2}})'dt=-\frac{dt}{2t^{\frac{3}{2}}},v=-\cos t\end{array}\right|=-\frac{\cos t}{\sqrt{t}}\Bigg|_{\frac{\pi}{2}}^{\infty}-\frac{1}{2}\int_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt=-\frac{1}{2}\int_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt.$$

Barcha $t\geq\frac{\pi}{2}$ uchun $\left|\frac{\cos t}{t^{\frac{3}{2}}}\right|\leq\frac{1}{t^{\frac{3}{2}}}$ bo'lib $\int_{\frac{\pi}{2}}^{\infty}\frac{dt}{t^{\frac{3}{2}}}$ integral yaqinlashganligi sababli $\int_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt$ integral absolyut yaqinlashadi.

Shunday qilib $\int_0^{\infty}\sin x^2dx$ xosmas integral aniq integral bilan yaqinlashuvchi xosmas integralning yig'indisidan iborat bo'lganligi uchun u yaqinlashadi.

$\int_0^{\infty}\cos x^2dx$ xosmas integralning yaqinlashuvchanligi ham shunga o'xshash isbotlanadi.

$\int_0^{\infty}\sin x^2dx$, $\int_0^{\infty}\cos x^2dx$ integrallar Freneli integrallari deb ataladi va yorug'likning difraksiyasi nazariyasida uchraydi.

Ikkinchi tur xosmas integral uchun ham 39.1 va 39.2 teoremaga o'xshash teoremalar mavjud.

39.1′-teorema. *f(x)* va *φ(x)* funksiyalar *(a,b]* oraliqda uzluksiz bo'lib $x=a$ nuqta ularning ikkinchi tur uzilish nuqtasi (maxsus nuqtasi) bo'lsin. Agar *(a,b]* oraliqning barcha nuqtalarida

$$0 \leq f(x) \leq \varphi(x)$$

tengsizlik bajarilsa, u holda: *a)* $\int_a^b \varphi(x)dx$ xosmas integral yaqinlashsa $\int_a^b f(x)dx$ xosmas integral ham yaqinlashadi.

b) $\int_a^b f(x)dx$ xosmas integral uzoqlashsa $\int_a^b \varphi(x)dx$ xosmas integral ham uzoqlashadi.

39.2′-teorema. *a* nuqta *(a,b]* oraliqda uzluksiz *f(x)* funksiyaning maxsus nuqtasi bo'lib $\int_a^b |f(x)|dx$ xosmas integral yaqinlashsa $\int_a^b f(x)dx$ xosmas integral ham yaqinlashadi.

13-misol. $\int_0^1 \frac{dx}{\sqrt[3]{x+3x^3}}$ xosmas integral tekshirilsin. $\int_a^b f(x)dx$ yaqinlashuvchi bo'lib $\int_a^b |f(x)|dx$ uzoqlashsa $\int_a^b f(x)dx$ shartli yaqinlashadi deyiladi.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x}+3x^2}$ funksiya *(0,1]* oraliqda uzluksiz bo'lib u $x=0$ nuqtada ikkinchi tur uzilishga ega. *(0,1]* oraliqdagi barcha *x* lar uchun

$$\frac{1}{\sqrt[3]{x}+3x^2} < \frac{1}{\sqrt[3]{x}}$$

tengsizlik bajarilib $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ xosmas integral yaqinlashgani uchun (4-misolga qarang) 39.1′-teoremaning *a)* bandiga binoan qaralayotgan xosmas integral ham yaqinlashadi.

14-misol. $\int_1^2 \frac{2+\sin x}{(x-1)^2}dx$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{2+\sin x}{(x-1)^2}$ funksiya *(1,2]* oraliqda uzluksiz bo'lib u $x=1$ nuqtada ikkinchi tur uzilishga ega va uning surati x ning istalgan qiymatida $2+sinx\geq 1$, chunki $sinx\geq -1$.

Shuning uchun. $\frac{2+\sin x}{(x-1)^2}\geq \frac{1}{(x-1)^2}$. Biroq $\int_1^2 \frac{dx}{(x-1)^2}$ uzoqlashadi, chunki $\alpha=2>1$ (4-misol). Demak 39.1′ teoremaning *b)* bandiga ko'ra berilgan integral ham uzoqlashadi.

15-misol. $\int_0^1 \frac{\cos x dx}{\sqrt{x}}$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\cos x}{\sqrt{x}}$ funksiya $x=0$ maxsus nuqtaga ega. $cosx$ funksiya $[0,\pi]$ kesmada ishorasini saqlamaydi, ya'ni u $[0,\frac{\pi}{2})$ da musbat, $(\frac{\pi}{2},\pi]$ da manfiy. Shuning uchun

$$\int_0^\pi \left|\frac{\cos x}{\sqrt{x}}\right|dx$$

Integralni qaraymiz. Barcha x lar uchun $|cosx|\leq 1$ ekanini hisobga olsak $[0,\pi]$ oraliqdagi barcha x lar uchun

$$\left|\frac{\cos x}{\sqrt{x}}\right|\leq\frac{1}{\sqrt{x}}$$

ekanligi kelib chiqadi.

$\int_0^1 \frac{dx}{\sqrt{x}}$ xosmas integral yaqinlashganligi (4-misol) uchun 39.1'-teoremaning *a)* bandiga binoan

$$\int_0^\pi \left|\frac{\cos x}{\sqrt{x}}\right|dx$$

xosmas integral ham yaqinlashadi.

39.2′-teoremaga ko'ra berilgan xosmas integral ham yaqinlashadi. Demak u absolyut yaqinlashadi.