Mavzu: Aniq integral tushunchasi

Reja.

1. Aniq integral tushunchasiga keltiruvchi yuza haqidagi masala.
2. Aniq integralning ta'rifi. Integrallanuvchi funksiyalar sinfi.

Tayanch iboarlar: Figuraning yuzi, egri chiziqli trapetsiya, integral yig'indisi, bo'linish odimi, aniq integral, intgrallanuvchi. Dirixle funksiyasi, aniq integralning geometrik ma'nosi.

Aniq integral tushunchasiga keltiruvchi yuza haqidagi masala

Maktab geometriya kursida kesmalar bilan chegaralangan figuralarning yuzlarini, doira hamda uning bo'lagini yuzini topish o'rganiladi. Shuningdek, egri chiziqlar bilan chegaralangan figuraning yuzini topish ham qisman o'rganiladi.

Bu yerda ixtiyoriy yopiq egri chiziq bilan chegaralangan yassi figuraning yuzini topish masalasi bilan jiddiy shug'ullanamiz.

Avvaliga xususiy holni, ya'ni figura $o\ x\ y$ tekisligiga joylashgan bo'lib, yuqoridan uzluksiz $y=f(x)$ $(f(x)\geq 0)$ egri chiziq, quyidan $o\ x$ o'qning $[a,b]$ $(a<b)$ kesmasi va yon tomonlardan $x=a,$ $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan holni qaraymiz. Bu figurani egri chiziqli **trapetsiya** deb ataymiz va $[a,b]$ kesmani uning **asosi** deymiz. Shu egri chiziqli trapetsiyaning yuzini topamiz.

$[a,b]$ kesmani

$$a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$$

nuqtalar bilan n ta ixtiyoriy

$$[x_0 ,x_1], [x_1 ,x_2],\ldots [x_{n-1} ,x_n]$$

kismlarga ajratamiz. Bu nuqtalar orqali $o\ y$ ga parallel to'g'ri chiziqlar o'tkazib egri chiziqli trapetsiyani n ta kichik trapetsiyasalarga ajratamiz. U holda qaralayotgan egri chiziqli trapetsiyaning yuzi n ta kichik egri chiziqli trapetsiyachalarning yuzlari yig'indisiga teng bo'lishi ravshan.

Shuning uchun S orqali egri chiziqli trapetsiyaning yuzini ΔS_k orqali asosi $[x_{k-1}, x_k]$ bo'lgan kichik egri chiziqli trapetsiyaning yuzini belgilasak $S=\Delta S_1 +\Delta S_2 +\ldots+\Delta S_n$ bo'ladi.

Qisqacha buni $S=\sum_{k=1}^{n}\Delta S_k$ ko'rinishda yozish qabul qilingan.

$$[x_0 ,x_1], [x_1 ,x_2],\ldots [x_{n-1} ,x_n]$$

kesmalarning har birida bittadan ixtiyoriy nuqta olib ularni $z_1, z_2,\ldots,z_n$ lar orqali belgilaymiz. Bu abssissalarda egri chiziq nuqtalarining ordinatalari $f(z_1), f(z_2),\ldots ,f(z_n)$ larni yasaymiz.

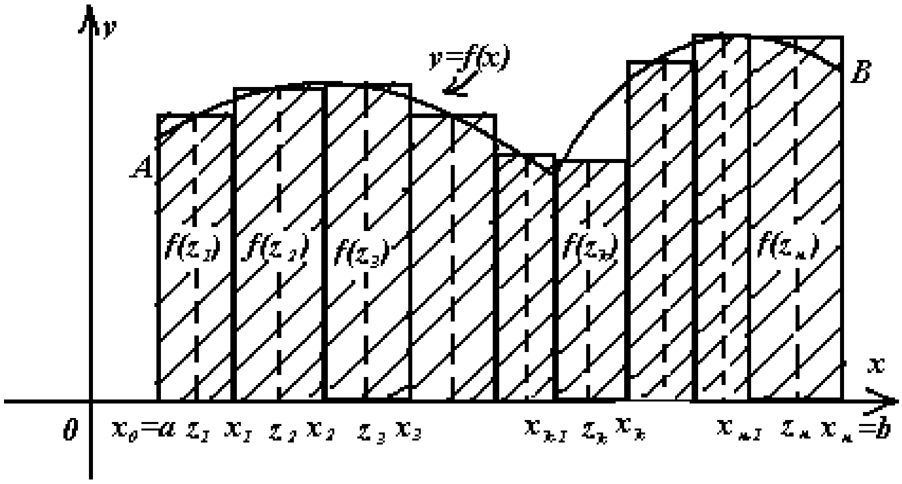
Keyin har bir asosi $[x_{k-1}, x_k]$ $(k=1,2, \ldots n)$ bo'lgan egri chiziqli trapetsiyachalar yuzini, asosi xuddi shunday, balandligi $f(z_k)$ bo'lgan to'g'ri to'rtburchakning yuzi bilan almashtiramiz. Bu to'g'ri to'rtburchakning yuzi

$$f(z_k)\,\Delta x_k$$

bo'lishi ravshan, bunda $\Delta x_k =x_k -x_{k-1}$ $[x_{k-1} ,x_k]$ kesmaning uzunligi (Δx_k -asos, $f(z_k)$ -balandlik).

Bu yuzani mos egri chiziqli tarpetsiyaning yuzini taqribiy qiymati deb qabul qilsak $\Delta S_k \approx f(z_k)\,\Delta x_k$ va $S \approx f(z_1)\,\Delta x_1 + f(z_2)\,\Delta x_2 + \ldots + f(z_n)\,\Delta x_n$ yoki qisqacha

$$S = \sum_{k=1}^{n} f(z_k)\Delta x_k \quad (38.1)$$ bo'ladi.



1-.shizma

Agar λ orqali $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ larning eng kattasini beligalasak λ kichrayganda (38.1) formulaning aniqlik darajasi ortadi. Shuning uchun (38.1) ning o'ng tomonidagi ifodaning $\lambda \to 0$ dagi limitini S ning **aniq qiymati** deb qabul qilish mumkin, ya'ni

$$S = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(z_k)\Delta x_k . \qquad (38.2)$$

Shunday qilib, egri chiziqli trapetsiyaning yuzini topish masalasi (38.2) ko'rinishdagi yig'indining limitini topishga olib keldi.

Ixtiyoriy yopiq egri chiziq bilan chegaralangan yassi figuraning yuzini topish masalasi ham egri chiziqli trapetsiyaning yuzini topish masalasiga keltirilishini ta'kidlab o'tamiz.

Yuzadan tashqari ko'pgina masalalarning yechimi ham (38.2) ko'rinishdagi limitni topishga kelishini ta'kidlab o'tamiz. Shuning uchun $\sum_{k=1}^{n} f(z_k)\Delta x_k$ ko'rinishdagi yig'indi mazmunan nimani anglatishidan qat'iy nazar uning limintini o'rganamiz.

Aniq integralning ta'rifi. Integrallanuvchi funksiyalar sinfi

Aniq integral oliy matematikaning eng muhim tushunchalaridan biridir. Yuzlarni, yoylarning uzunliklarini, hajmlarni, ishni, inersiya va statik momentlarni, og'irlik markazi koordinatalarini, yo'lni, bosimni va hokazolarni uning yordamida hisoblash mumkin.

$[a,b]$ kesmada aniqlangan $y=f(x)$ funksiya berilgan bo'lsin. Quyidagi amallarni bajaramiz:

1) $[a,b]$ kesmani

$$a=x_0<x_1<x_2<\ldots<x_{k-1}<x_k<\ldots<x_{n-1}<x_n=b$$

bo'luvchi nuqtalar yordamida n ta «kichik»

$$[x_0,x_1],\ [x_1,x_2], \ldots,\ [x_{k-1},x_k], \ldots,\ [x_{n-1},x_n]$$

kesmalarga ajratib ularning uzunliklarini mos ravishda Δx_1 , Δx_2 ,..., Δx_n lar orqali belgialymiz.

2) Har bir $[x_{k-1},x_k]$ $(k=1,2,\dots,n)$ kesmada bittadan ixtiyoriy nuqta tanlab olib ularni mos ravishda $z_1, z_2,\dots,z_n$ lar orqali belgilaymiz.

3) Tanlangan nuqtalarda funksiyaning

$f(z_1), f(z_2),\dots, f(z_n)$

qiymatlarini hisoblaymiz.

4) Funksiyaning hisoblangan qiymatini tegishli $[x_{k-1}, x_k]$ kesmachaning uzunligi Δx_k ga ko'paytirib $f(z_1)\,\Delta x_1$, $f(z_2)\,\Delta x_2,\dots$, $f(z_k)\Delta x_k,\dots$, $f(z_n)\Delta x_n$ ko'paytmalarni tuzamiz.

5) Tuzilgan ko'paytmalarni qo'shamiz va yig'indini $\boldsymbol{\sigma_n}$ bilan belgilaymiz:

$$\sigma_n= f(z_1)\,\Delta x_1+ f(z_2)\,\Delta x_2,+\dots+ f(z_k)\Delta x_k+\dots+ f(z_n)\Delta x_n$$

yoki qisqacha $\sigma_n=\sum_{k=1}^{n} f(z_k)\Delta x_k$.

$\boldsymbol{\sigma_n}$ yig'indi $f(x)$ funksiya uchun $[a,b]$ kesmada tuzilgan **integral yig'indi** deb ataladi.

6) $[x_{k-1}, x_k]$ kesmalarning uzunliklaridan eng kattasini λ orqali belgilab uni **bo'linish odimi** deb ataymiz.

Endi bo'linishlar soni n ni orttira boramiz $(n\to\infty)$ va bunda $\lambda\to 0$di deb faraz qilamiz.

1-ta'rif. Agar $\lambda \to 0$ da σ_n integral yig'indisi $[a,b]$ kesmani qismiy $[x_{k-1}, x_k]$ kesmalarga ajratish usuliga va ularning har biridan z_k nuqtani tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son $[a,b]$ kesmada $f(x)$ funksiyadan olingan **aniq integral** deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi.

Bu yerda $f(x)$-integral ostidagi funksiya $[a,b]$-kesma integrallash oralig'i, a va b sonlar integrallashning quyi va yuqori chegarasi. x-integrallash o'zgaruvchisi deyiladi.

Shunday qilib, aniq integralning ta'rifidan $\int_a^b f(x)dx=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k$ kelib chiqadi.

Aniq integralning ta'rifidan aniq integral mavjud bo'lishi uchun $f(x)$ funksiyaning $[a,b]$ kesmada chegeralangan bo'lishi zarur.

2-ta'rif. Agar chekli $\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k$ mavjud bo'lsa $f(x)$ funksiya $[a,b]$ kesmada **integrallashuvchi** deyiladi.

1-teorema (funksiya integrallashuvchi bo'lishining zaruriy sharti). Agar $f(x)$ funksiya $[a,b]$ kesmada integrallashuvchi bo'lsa, u shu kesmada chegaralangandir.

Isboti. Teskarisini faraz qilamiz, ya'ni $[a,b]$ da integrallashuvchi $f(x)$ funksiya shu kesmada chegaralanmagan bo'lsin. U holda σ_n integral yig'indini $z_1, z_2, z_3,..,z_n$ nuqtalarni tanlash hisobiga istalgancha katta qilish mumkinligini, ya'ni $\lim_{\lambda\to 0}\sigma_n$ limit mavjud bo'lmasligini ko'rsatamiz.

Bu holda *[a,b]* kesmani istalgan $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$,…, $[x_{n-1},x_n]$ kesmalarga ajratilishini qaramaylik *f(x)* funksiya *[a,b]* kesmada chegaralanmaganligi sababli u shu kesmalarning kamida bittasi, masalan $[x_0,x_1]$da chegaralanmagan bo'ladi. Qolgan $[x_1,x_2]$, $[x_2,x_3]$,…, $[x_{n-1},x_n]$ kesmalarda bittadan ixtiyoriy nuqtalarni olib ularni mos ravishda $z_2, z_3,..,z_n$ lar orqali belgilaymiz va $f(z_2)\Delta x_2 + f(z_3)\Delta x_3 + \ldots + f(z_n)\Delta x_n = \delta_n'$
deb olamiz. *f(x)* funksiya $[x_0,x_1]$ kesmada chegarlanmaganligi uchun istalgan katta $M>0$ sonni olmaylik $[x_0,x_1]$ kesmada Shunday z_1 nuqta mavjud bo'lib,

$$|f(z_1)| \geq \frac{|\sigma_n'| + M}{\Delta x_1} \quad \text{yoki} \quad |f(z_1)|\Delta x_1 \geq |\sigma_n'| + M$$

bo'ladi. Ikkinchi tomondan

$$\sigma_n = f(z_1)\Delta x_1 + f(z_2)\Delta x_2 + \ldots + f(z_k)\Delta x_k + \ldots + f(z_n)\Delta x_n = f(z_1)\Delta x_1 + \sigma_n'$$

bo'lgani uchun $|\sigma_n| = |f(z_1)\Delta x_1 + \sigma_n'| \geq |f(z_1)|\Delta x_1 - |\sigma_n'| \geq |\sigma_n'| + M - |\sigma_n'| = M$
kelib chiqadi. $|\sigma_n| \geq M$ tengsizlik $\lim\limits_{\lambda\to 0}\sigma_n$ ning mavjud emasligini, ya'ni *f(x)* funksiyaning *[a,b]* kesmada integrallanuvchi emasligini ko'rsatadi. Bu teoremaning shartiga zid. Bu ziddiyatga *[a,b]* kesmada integrallanuvchi *f(x)* funksiya shu kesmada chegaralanmagan deb qabul qilgan noto'g'ri farazimiz oqibatida keldik.

Keltirilgan teorema faqatgina zaruriy shartdan iborat bo'lib, u yetarli emas, ya'ni funksiyaning chegaralanganligidan uning integrallanuvchiligi kelib chiqmaydi.

Masalan, Dirixle funksiyasi

$$f(x) = \begin{cases} 1, & agar \quad x \quad ratsional \quad bo'lsa, \\ 0, & agar \quad x \quad irratsional \quad bo'lsa \end{cases}$$

ni *[0,1]* kesmada qarasak u shu kesmada chegarlangan$(|f(x)| \leq 1)$. Ammo bu funksiya *[0,1]* kesmada integrallanuvchi emas.

Haqiqatan, agar *[0,1]* kesmani kichik kismlarga ajratilganda $z_1, z_2, z_3,..,z_n$ nuqtalar sifatida ratsional sonlar olinsa integral yig'indi

$$\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k = \sum_{k=1}^{n} 1\cdot\Delta x_k = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n = 1$$

bo'ladi va $z_1, z_2, z_3,..,z_n$ lar sifatida irratsional sonlar olinsa integral yig'indi

$$\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k = \sum_{k=1}^{n} 0\cdot\Delta x_k = 0$$

bo'ladi. Bu tengliklardan σ_n integral yig'indi $\lambda\to 0$ da limitga ega emasligi, ya'ni Dirixle funksiyasi *[0,1]* kesmada integrallanuvchi emasligi kelib chiqadi.

Bu misol shuni ko'rsatadiki, hatto chegaralangan funksiyalarning ham aniq integrallari mavjud bo'lmasligi mumkin ekan. Qanaqa funksiyalarning aniq integrallari har doim mavjud bo'ladi degan savolga quyidagi teoremalar javob beradi. Biz ularni isbotsiz keltiramiz.

2-teorema. Agar funksiya *[a,b]* kesmada uzluksiz bo'lsa, u shu kesmada integrallanuvchidir.

3 teorema. Agar funksiya *[a,b]* kesmada chegaralangan va shu kesmaning chekli sondagi nuqtalarida uzilishga ega bo'lib, qolgan barcha nuqtalarda uzluksiz bo'lsa, funksiya *[a,b]* da integrallanuvchidir.

Natija. *[a,b]* kesmaning chekli sondagi nuqtalaridagina teng bo'lmagan ikkita *f(x)* va *φ(x)* funksiyalardan biri shu kesmada integrallanuvchi bo'lsa, u holda ikkinchisi ham integrallanuvchi bo'lib ularning integrallari teng bo'ladi.

4-teorema. Agar *f(x)* funksiya *[a,b]* kesmada chegaralangan va monoton bo'lsa, u shu kesmada integrallanuvchidir.

[a,b] kesmaning cheksiz ko'p nuqtalarda uzilishga ega bo'lgan chegarlangan funksiyalar orasida integrallanuvchi bo'lganlari ham bo'lmaganligi ham mavjud. Integrallanuvchi bo'lmaganiga Dirixle funksiyani misol keltirish mumkin.

Chegarlangan va cheksiz ko'p nuqtalarda uzilishga ega bo'lib integrallanuvchi funksiyaga

$$f(x)=\begin{cases} 1, agar\ \dfrac{1}{2n}<x\le\dfrac{1}{2n-1}\ bo'lsa, \\ -1, agar\ \dfrac{1}{2n+1}<x\le\dfrac{1}{2n}\ bo'lsa, n=1,2,3,... \\ 0, agar\ \quad x=0\quad bo'lsa \end{cases}$$

funksiyani misol keltirish mumkin. Bu funksiya *[a,b]* kesmada chegaralangan va barcha $x_n=\frac{1}{n}, n=1,2,3,...$ nuqtalarda birinchi tur uzilishga ega.

Endi aniq integralning ta'rifidan bevosita kelib chiqadigan xossalarini keltiramiz.

1-izoh. Aniq integralning qiymati integrallash o'zgaruvchisining qanday harf bilan belgilanishiga bog'liq emas. Masalan: $\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(z)dz.$

2-izoh. Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

3-izoh. Chegaralari teng bo'lgan aniq integralning qiymati *0* ga teng. $\int_a^a f(x)dx=0.$

Aniq integralning ta'rifidan foydalanib egri chiziqli trapetsiyaning yuzini ifodalovchi (38.2) tenglikni $S=\int_a^b f(x)dx$

ko'rinishda yozish mumkin. Boshqacha so'z bilan aytganda $\int_a^b f(x)dx$ aniq integralning **geometrik ma'nosi** egri chiziqli trapetsiyaning **yuzini** ifodalar ekan.

Aniq integralning asosiy xossalari. Aniq integralni hisoblash

. Aniq integralning asosiy xossalari.
. Integralning yuqori chegarasi bo'yicha hosilasi.
. Aniq integralni hisoblash. Nyuton-Leybnis formulasi.
. Aniq integralda o'zgaruvchini almashtirish.
Aniq integralni bo'laklab integrallash.

Aniq integralning asosiy xossalari

1-xossa. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya'ni *A=cons t* bo'lsa

$$\int_a^b Af(x)dx = A\int_a^b f(x)dx$$

bo'ladi, bunda *f(x)* integrallashuvchi funksiya.

Isboti.

$$\int_a^b Af(x)dx = \lim_{\lambda\to 0}\sum_{k=1}^{n} Af(z_k)\Delta x_k = A\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k = A\int_a^b f(x)dx.$$

2-xossa. Bir nechta integrallashuvchi funksiyalarning algebraik yig'indisining aniq integrali qo'shiluvchilar integrallarining yig'indisiga teng, ya'ni

$$\int_a^b [f(x)\pm\varphi(x)]dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx\,.$$

Isboti.

$$\int_a^b [f(x)\pm\varphi(x)]dx = \lim_{\lambda\to 0}\sum_{k=1}^{n}[f(z_k)\pm\varphi(z_k)]\Delta x_k = \lim_{\lambda\to 0}\left[\sum_{k=1}^{n} f(z_k)\Delta x_k \pm \sum_{k=1}^{n}\varphi(z_k)\Delta x_k\right] =$$

$$= \lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k \pm \lim_{\lambda\to 0}\sum_{k=1}^{n}\varphi(z_k)\Delta x_k = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$$

3-xossa. Agar quyidagi uch integralning har biri mavjud bo'lsa, u holda har qanday uchta *a,b,c* sonlar uchun

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (1)$$

tenglik o'rinli bo'ladi.

Isboti. Dastlab *a<c< b* deb faraz qilib *f(x)* funksiya uchun *[a,b]* kesmada integral yig'indi σ_n ni tuzamiz. Integral yig'indining limiti *[a,b]* kesmani bo'laklarga bo'lish usuliga bog'liq bo'lmagani uchun *[a,b]* kesmani mayda kesmachalarga shunday bo'lamizki, *c* nuqta bo'lish nuqtasi bo'lsin.

Agar, masalan, $c = x_m$ bo'lsa, u holda σ_n integral yig'indini ikkita yig'indiga ajratamiz:

$$\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k = \sum_{k=1}^{m} f(z_k)\Delta x_k + \sum_{k=m+1}^{n} f(z_k)\Delta x_k\,.$$

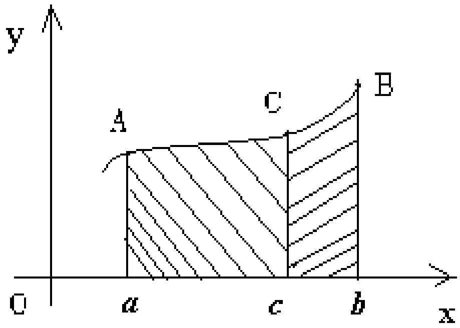
Ushbu tenglikda $\lambda\to 0$da limitga o'tsak isbotlanishi lozim bo'lgan (39.1) kelib chiqadi.

$a< b<c$ bo'lsin. U holda isbotlanganga muvofiq

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$ bo'ladi.

Bundan $$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

ya'ni (39.1) ga ega bo'ldik.



2-chizma.

141-chizmada *f(x)>0* va *a<c< b* bo'lgan hol uchun 3-xossaning geometrik tasviri berilgan: *a A B b* egri chiziqli trapetsiyaning yuzi *a A C c* va *c C B b* egri chiziqli trapetsiyalar yuzlarini yig'indisiga teng.

4-xossa. Agar *[a,b]* kesmada *f (x)* funksiya integrallanuvchi va *f (x)≥0* bo'lsa, u holda

$$\int_a^b f(x)dx \ge 0$$ bo'ladi.

Isboti. Istalgan *k* uchun $f(x_k)\ge 0$, $\Delta x_k > 0$ bo'lgani sababli $\sum_{k=1}^{n} f(x_k)\Delta x_k \ge 0$ bo'ladi. Bunda $\lambda \to 0$ da limitga o'tsak isbotlanishi lozim bo'lgan tengsizlikni hosil qilamiz.

Shuningdek *[a,b]* kesmada *f (x)≤0* bo'lganda $\int_a^b f(x)dx \le 0$ bo'lishini ko'rsatish qiyin emas.

5-xossa. Agar *[a,b]* (*a<b*) kesmada ikkita integrallanuvchi *f (x) va φ(x)* funksiya *f (x) ≥ φ(x)* shartni qanoatlantirsa,u holda

$$\int_a^b f(x)dx \ge \int_a^b \varphi(x)dx$$

tengsizlik o'rinli.

Isboti. *[a,b]* da *f (x)- φ(x) ≥ 0* bo'lgani uchun 4- xossaga ko'ra

$$\int_a^b [f(x) - \varphi(x)]dx \ge 0$$ bo'ladi. Bundan 2-xossasiga binoan

$$\int_a^b f(x)dx - \int_a^b f(x)dx \ge 0 \qquad \text{yoki} \qquad \int_a^b f(x)dx \ge \int_a^b f(x)dx$$

kelib chiqadi.

6- xossa. Agar *f(x)* va | *f(x)*| funksiya *[a,b]* da integrallanuvchi bo'lsa, u holda

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \quad (2)$$

tengsizlik o’rinli.

Isboti. -|*f(x)*|≤*f(x)*≤|*f(x)*| ga 5- xossani qo’llasak

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

yoki

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

tengsizlik hosil bo’ladi.

Natija. Agar *[a,b]* kesmada *f(x)* va | *f(x)*| funksiya integrallanuvchi bo’lib, shu kesmada | *f(x)*| ≤ *k* *(k=const)* bo’lsa, u holda

$$\left|\int_a^b f(x)dx\right| \le k(b-a) \qquad (3)$$

tengsizlik o’rinli.

Haqiqatan, | *f(x)*| ≤ *k* bo’lgani uchun 6-5 va 1-xossaga asosan

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \le k\int_a^b dx$$

bo’ladi. Bunda

$$\int_a^b dx = \ell im_{\lambda\to 0}\sum_{k=1}^{n} 1\cdot \Delta x_k = \Delta x_1 + \Delta x_2 + ... + \Delta x_n = b-a$$

ekanini hisobga olsak (39.3) tengsizlikka ega bo’lamiz.

7- xossa. (Aniq integralni baholash). Agar *m* va *M* sonlar *[a,b]* kesmada integrallanuvchi *f(x)* funksiyaning eng kichik va eng katta qiymati bo’lsa, u holda

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a) \qquad (4)$$

tengsizlik o’rinli.

Isboti. Shartga binoan *[a,b]* kesmada barcha *x* lar uchun *m* ≤ *f(x)* ≤ *M.*
Bunga 5- xossani qo’llasak

$m\int_a^b dx \le \int_a^b f(x)dx \le M\int_a^b dx$ yoki $\int_a^b dx = b-a$ ekanini hisobga olsak oxirgi tengsizliklardan (39.4) ga ega bo’lamiz

8- xossa. Agar *f(x)* funksiya *[a,b]* kesmada integrallanuvchi bo’lib *m* va *M* uning shu kesmadagi eng kichik va eng katta qiymati bo’lsa, u holda shunday o’zgarmas μ

$(m \le \mu \le M)$ son mavjudki

$$\int_a^b f(x)dx = \mu\cdot(b-a) \qquad (5)$$

tenglik o’rinli.

Isboti. (39.4) ni *в-a* ga bo'lsak $m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$ bo'ladi.

$\frac{1}{b-a}\int_a^b f(x)dx = \mu$

belgisini kiritamiz. U holda oxirgi tenglikni *b-a* ga ko'paytirib isbotlanishi lozim bo'lgan (5) tenglikka ega bo'lamiz.

Natija (o'rta qiymat haqidagi teorema). Agar *f(x)* *[a,b]* kesmada uzluksiz funksiya bo'lsa, u holda kesmada shunday *x=c* nuqta topiladiki, bu nuqtada

$$\int_a^b f(x)dx = f(c)(b-a) \qquad (6)$$

tenglik o'rinli.

Haqiqatan. *f(x)* funksiya *[a,b]* kesmada uzluksiz bo'lganligi tufayli u shu kesmada o'zining eng kichik *m* va eng katta *M* qiymatini qabul qiladi. Uzluksiz funksiya *[m,M]* kesmadagi barcha qiymatlarni qabul qilganligi sababli u $\mu = \frac{1}{b-a}\int_a^b f(x)dx$ qiymatni ham qabul qiladi, ya'ni *[a,b]* kesmada shunday *x=c* nuqta mavjud bo'lib *f(c)*= μ bo'ladi. (5) tenglikka μ o'rniga *f(c)* ni qo'yib isbotlanishi lozim bo'lgan (39.6) tenglikni hosil qilamiz.

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ qiymat *f(x)* funksiyaning *[a,b]* kesmadagi **o'rtacha qiymati** deb ataladi

Bu natijaga quyidagicha geometrik izoh berish mumkin. *[a,b]* kesmada *f (x)*≥ 0 bo'lganda aniq integralning qiymati asosi *b-a* va balandligi *f(c)* bo'lgan to'g'ri to'rtburchakning yuziga teng bo'lar ekan.

Agar *f(x)* va *g(x)* funksiyalar *[a,b]* kesmada integrallanuvchi bo'lsa, u holda ularning ko'paytmasi *f(x)·g(x)* ham shu kesmada integrallashuvchi bo'lishini ta'kidlab o'tamiz.

Integralning yuqori chegarasi bo'yicha hosilasi

Agar aniq integralda integrallashning quyi chegarasi *a* ni aniq qilib belgilansa va yuqori chegarasi *x* esa o'zgaruvchi bo'lsa, u holda integralning qiymati ham *x* o'zgaruvchining funksiyasi bo'ladi.

Quyi chegarsi *a* o'zgarmas bo'lib yuqori chegarasi *x* o'zgaruvchi bo'lgan $\int_a^x f(t)dt$ *(a≤x≤b)* integralni qaraymiz. Bu integral yuqori chegara *x* ning funksiyasi bo'lganligi sababli uni ϕ *(x)* orqali belgilaymiz, ya'ni

$$\phi(x) = \int_a^x f(t)dt$$

va uni yuqori chegarsi o'zgaruvchi integral deb ataymiz.

4-teorema. Agar *f(x)* funksiya *[a,b]* kesmada uzluksiz bo'lsa, u holda

$$\phi'(x) = \left(\int_a^x f(t)dt \right)' = f(x)$$

tenglik o'rinli.

Isboti. *[a,b]* ga tegishli istalgan *x* ni olib unga shunday $\Delta x \neq 0$ orttirma beramizki $x + \Delta x$ ham *[a,b]* ga tegishli bo'lsin. U holda $\phi(x)$ funksiya

$$\phi(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt$$

yangi qiymatni qabul qilinadi. Aniq integralning 3-xossasiga ko'ra

$$\phi(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt = \phi(x) + \int_x^{x+\Delta x} f(t)dt$$

bo'ladi. Demak, $\phi(x)$ funksiyaning orttirmasi

$$\Delta\varphi(x) = \varphi(x + \Delta x) - \varphi(x) = \int_x^{x+\Delta x} f(t)dt$$

bo'ladi.

Oxirgi tenglikka o'rta qiymat haqidagi teoremani qo'llasak

$$\Delta\phi(x) = f(c)(x + \Delta x - x) = f(c)\Delta x$$

hosil bo'ladi, bunda *c* *x* bilan $x + \Delta x$ orasidagi son. Tenglikni har ikkala tomonini Δx ga bo'lamiz:
$$\frac{\Delta\phi(x)}{\Delta x} = f(c)$$

Agar Δx *0* ga intilsa *c* *x* ga intiladi va *f(x)* funksiyaning *[a,b]* kesmada uzluksizligidan *f(c)* ning *f(x)* ga intilishi kelib chiqadi.

Shuning uchun oxirgi tenglikda $\Delta x \to 0$ da limitga o'tib quyidagini hosil qilamiz:

$$\phi'(x) = \lim_{\Delta x \to 0} \frac{\Delta\phi(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \lim_{\Delta x \to 0} f(c) = \lim_{c \to 0} f(c) = f(x)$$

Bu teoremaga binoan *[a,b]* kesmada uzluksiz *f(x)* funksiya boshlang'ich funksiyaga ega ekanligi va $\phi(x) = \int_a^x f(t)dt$ shu funksiyaning boshlang'ich funksiyalaridan biri bo'lishi kelib chiqadi.

Agar *f(x)* ning boshqa boshlang'ich funksiyalari uning $\phi(x)$ boshlang'ich funksiyasidan faqatgina o'zgarmas *C* songa farq qilishini hisobga olsak, aniqmas va aniq integrallar orasida bog'lanish o'rnatuvchi

$$\int f(x)dx = \int_a^x f(t)dt + C$$

tenglikka ega bo'lamiz.

Aniq integralni hisoblash. Nyuton-Leybnis formulasi

Aniq integrallarni integral yig'indining limiti sifatida bevosita hisoblash ko'p hollarda juda qiyin, uzoq hisoblashlarni talab qiladi va amalda juda kam qo'llaniladi. Aniq integralni hisoblash uchun Nyuton-Leybnis formulasini kashf etilishi aniq integralni qo'llanish ko'lamini kengayishiga asosiy sabab bo'ldi.

5-teorema. Agar *F(x)* funksiya uzluksiz *f(x)* funksiyaning *[a,b]* kesmadagi boshlang'ich funksiyasi bo'lsa, u holda $\int_a^x f(x)dx$ aniq integral boshlang'ich funksiyaning integrallash oraligidagi orttirmasiga teng, ya'ni $\int_a^в f(x)dt = F(b) - F(a)$. (7) tenglik aniq integralni hisoblashning **aosiy formulasi** yoki **Nyuton-Leybnis formulasi** deyiladi.

Isboti. Shartga ko'ra *F(x)* funksiya *f(x)* ning biror boshlang'ich funksiyasi bo'lsin. $\phi(x) = \int_a^x f(t)dt$ funksiya ham *f(x)* ning boshlang'ich funksiyasi bo'lganligi uchun $\phi(x)=F(x)+C$ yoki $\int_a^x f(t)dt = F(x) + C.$ $x=a$ desak $\int_a^x f(t)dt = F(a) + C$, $0=F(a)+C, C=-F(a).$

Demak, $\int_a^x f(t)dt = F(x) - F(a)$.

Endi $x=b$ desak, Nyuton-Leybnis formulasini hosil qilamiz:

$$\int_a^в f(t)dt = F(b) - F(a).$$

$F(b) - F(a) = F(x)\Big|_a^b$ belgilash kiritilsa Nyuton-Leybnis formulasi

$$\int_a^b f(x)dx = F(x)\Big|_a^b \qquad (8)$$

ko'rinishga ega bo'ladi.

1-misol. Integralni hisoblang: $\int_0^{\frac{\pi}{2}} \sin xdx$.

Yechish. *(-cosx)'=sinx* bo'lgani uchun

$$\int_0^{\frac{\pi}{2}} \sin x = -\cos x \Big|_0^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1) = 1.$$

2-misol. $\int_a^b x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_a^b = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1} \quad (\alpha \neq -1)$

3-misol. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\sin^2 x} = -ctgx \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\left(ctg\frac{\pi}{4} - ctg\frac{\pi}{6}\right) = -(1-\sqrt{3}) = \sqrt{3} - 1$

Shunday qilib *[a,b]* kesmada uzluksiz *f(x)* funksiya uchun $\int f(x)dx = F(x) + C$ bo'lganda $\int_a^b f(x)dx = F(x) + C = F(x)\Big|_a^b$ bo'lar ekan.

Aniq integralda o'zgaruvchini almashtiris

$\int_a^b f(x)dx$ integralni hisoblash talab etilsin, bunda *f(x)* funksiya *[a,b]* kesmada uzluksiz. $x=\varphi(t)$ almashtirish olamiz, bunda $\varphi(t)$ $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz $\varphi'(t)$ hosilaga ega hamda $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo'lsin. U holda

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

formula o'rinli bo'ladi.

Haqiqatan ham, agar *F(x)* funksiya *f(x)* ning boshlang'ich funksiyasi bo'lsa, u holda $F(\varphi(t))$ funksiya $f(\varphi(t))\,\varphi'(t)$ funksiya uchun boshlang'ich funksiya bo'lishi isbotlangan edi. Nyuton-Leybnis formulasiga ko'ra

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a);$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F(\varphi(t))\Big|_\alpha^\beta = F(\varphi(\beta))-F(\varphi(\alpha)) = F(b)-F(a)$$

4-misol. $\int_0^1 \sqrt{1-x^2}\,dx$ hisoblansin.

Yechish. $x=\sin t$ deb almashtirsak, $dx=\cos t\,dt$, $1-x^2=\cos^2 t$ bo'ladi.

$x=0$ da $\sin t=0$, $t=0$, $x=1$ da $\sin t=1$, $t=\frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)dt = \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(0+\frac{\sin 0}{2}\right)\right] = \frac{\pi}{4}$$

Aniq igtegralni bo'laklab integrallash

Faraz qilaylik, *u(x)* va *v(x)* funksiyalar *[a,b]* kesmada differensiallanuvchi funksiyalar bo'lsin. U holda

$$(uv)'=u'v+uv'$$

bo'ladi, buni *a* dan *b* gacha integrallasak

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \quad \text{yoki} \quad \int_a^b d(uv) = \int_a^b vdu + \int_a^b udv, (uv)\Big|_a^b = \int_a^b vdu + \int_a^b udV,$$

bundan $\int_a^b udv = (uv)\Big|_a^b - \int_a^b vdu..$

Bu formula aniq integralni **bo'laklab integrallash** formulasi deyiladi.

5- misol. $\int_1^e \ln x dx$ hisoblansin.

Yechish.

$$\int_{1}^{e}\ell nxdx\left|\begin{matrix}u=\ell nx, dv=dx\\ du=(\ell n_x)'dx=\dfrac{1}{x}dx, v=x\end{matrix}\right|=\ell nx\cdot x\Big|_1^e-\int_1^e x\frac{dx}{x}=\ell ne\cdot e-\ell n1\cdot 1-x\Big|_1^e=e-(e-1)=e-e+1=1$$

6- misol. $\int_0^2 xe^{-x}dx\left|\begin{matrix}u=x, dv=e^{-x}dx\\ du=dx, v=\int e^{-x}dx=-e^{-x}\end{matrix}\right|=x\cdot(-e^{-x})\Big|_0^2+\int_0^2 e^{-x}dx=$

$-(2e^{-2}-0\cdot e^{-0})-e^{-x}\Big|_0^2=-\dfrac{2}{e^2}-(e^{-2}-e^{-0})==\dfrac{2}{e^2}-\dfrac{1}{e^2}+1=1-\dfrac{3}{e^2}.$

7-misol.

$$\int_0^{\frac{\pi}{2}}x\cos xdx\left|\begin{matrix}u=x, \cos xdx=dv,\\ du=dx, v=\int\cos xdx=\sin x\end{matrix}\right|=x\cdot\sin x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin xdx=\frac{\pi}{2}\cdot\sin\frac{\pi}{2}-0\cdot\sin 0+\cos x\Big|_0^{\frac{\pi}{2}}=$$

$=\dfrac{\pi}{2}-0+\cos\dfrac{\pi}{2}-\cos 0=\dfrac{\pi}{2}-1.$

8-misol.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{xdx}{\sin^2 x}\left|\begin{matrix}u=x, dv=\dfrac{dx}{\sin^2 x}\\ du=dx, v=\int\dfrac{dx}{\sin^2 x}=-ctgx\end{matrix}\right|=x\cdot(-ctgx)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(-ctgx)dx=-\left(\frac{\pi}{3}\cdot ctg\frac{\pi}{3}-\frac{\pi}{4}ctg\frac{\pi}{4}\right)+$$

$$+\ell n\sin x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=-\left(\frac{\pi}{3}\cdot\frac{1}{\sqrt{3}}-\frac{\pi}{4}\cdot 1\right)+\ell n\sin\frac{\pi}{3}-\ell n\sin\frac{\pi}{4}=-\frac{\pi}{3\sqrt{3}}+\frac{\pi}{4}+\ell n\frac{\sqrt{3}}{2}-\ell n\frac{\sqrt{2}}{2}=$$

$$=\frac{\pi}{4}-\frac{\pi}{3\sqrt{3}}+\ell n\left(\frac{\sqrt{3}}{2}:\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}-\frac{\sqrt{3}\pi}{9}+\ell n\sqrt{\frac{3}{2}}=\frac{9\pi-4\sqrt{3}\pi}{36}+\frac{1}{2}\ell n\frac{3}{2}.$$

Mustaqil yechish uchun mashqlar

Funksiyalarning hosilasini toping.

1. $F(x)=\int_2^x e^{-3t}dt, x>2.$ Javob: ℓ^{-3x}.

2. $F(x)=\int_{x^2}^0\sin 2tdt$; $x>0$ Javob: $-3x^2\sin 2x^3$.

3. $f(x)=3-2\sin x$ funksiyaning $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ kesmadagi o'rtacha qiymatini toping. Javob: 3.

Integrallar baholansin.

4. $I=\int_0^2\sqrt{8-x^2}dx$. Javob: $4<I<4\sqrt{2}$.

5. $I=\int_0^{\pi}\dfrac{dx}{5-3\cos x}$. Javob: $\dfrac{\pi}{8}<I<\dfrac{\pi}{2}$.

Integrallarni Nyuton-Leybnis formulasidan foydalanib hisoblang.

6. $\int_{-1}^{1}(6x^2-2x-5)dx$ Javob: -6.

7. $\int_{0}^{1}\frac{dx}{(5-3x)^3}.$ Javob: $\frac{7}{200}$.

8. $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cdot\cos 3xdx.$ Javob: *0.*

9. $\int_{3}^{4}\frac{dx}{25-x^2}.$ Javob: $\frac{1}{5}\ell n\frac{3}{2}$

10. $\int_{3}^{4}\frac{dx}{x^2-2x+10}.$ Javob: $\frac{\pi}{12}$.

Integrallarni o'zgaruvchini almashtirib hisoblang.

11. $\int_{3}^{8}\frac{dx}{5-\sqrt{x+1}}.$ Javob: $10\ell n\frac{3}{2}-2$.

12. $\int_{3}^{8}\frac{dx}{\sqrt{(16+x^2)^3}}.$ Javob: $\frac{3}{80}$.

13. $\int_{\frac{\sqrt{2}}{2}}^{1}\frac{\sqrt{1-x^2}}{x^2}dx.$ Javob: $1-\frac{\pi}{4}$.

14. $\int_{0}^{\frac{\pi}{2}}\frac{dx}{5+4\cos x}.$ Javob: $\frac{2}{3}arctg\frac{1}{3}$.

Integrallarni bo'laklab integrallang.

15. $\int_{0}^{2}\frac{xdx}{e^{2x}}.$ Javob: $\frac{e^4-5}{4e^4}$.

16. $\int_{0}^{\pi}x\sin\frac{x}{2}dx$. Javob: *4.*

17. $\int_{1}^{\sqrt{e}}x\ell nxdx..$ Javob: $\frac{e+1}{4}$.

18. $\int_{0}^{\frac{\pi}{3}}\frac{xdx}{\cos^2 x}.$ Javob: $\frac{\pi}{3}-\ell n2$.

Mavzu: Xosmas integrallar

Aniq integral tushunchasini umumlashtirish.

Chegarasi cheksiz xosmas integrallar.

Chegaralanmagan funksiyaning xosmas integrallari.
Xosmas integrallarning yaqinlashish alomatlari.

Xosmas integral tushunchasini umumlashtirish

Aniq integralga ta‘rif berishda integrallash oralig’i *[a,b]* chekli va integral ostidagi funksiya shu oraliqda chegaralangan deb faraz qilgan edik. Ana shu shartlardan aqalli birortasi bajarilmasa aniq integralning keltirilgan ta‘rifi ma‘nosini yo’qotadi. Chunki integrallash oralig’i cheksiz bo’lganda uni uzunliklari chekli bo’lgan *n* ta qismga ajratib bo’lmaydi, integral ostidagi funksiya chegaralanmaganda integral yig’indi chekli limitga ega bo’lmaydi. Ammo aniq integral tushunchasini bu hollar uchun ham umumlashtirish mumkin. Umumlashtirish natijasida xosmas integrallar tushunchasiga kelamiz. Xosmas integrallar ikki turga chegaralari cheksiz xosmas integrallar hamda chegaralanmagan funkitsyaning xosmas integraliga bo’linadi.

Chegaralari cheksiz xosmas integrallar

1- ta‘rif. *f(x)* funksiya *[a,∞)* intervalda aniqlangan bo’lib, u istalgan chekli *[a,R] (R>a)* kesmada integrallanuvchi, ya‘ni $\int_a^R f(x)dx$ aniq integral mavjud bo’lsin. U holda

$$\lim_{R\to+\infty}\int_a^R f(x)dx \qquad (1)$$

chekli limit mavjud bo’lsa, u **birinchi tur** yoki **chegaralari cheksiz** xosmas integral deb ataladi, va

$$\int_a^{+\infty} f(x)dx \quad (2)$$

kabi belgilanadi.

Shunday qilib, ta‘rifga ko’ra

$$\int_a^{R} f(x)dx = \lim_{R\to+\infty}\int_a^R f(x)dx .$$

Bu holda (40.2.) xosmas integral **mavjud** yoki **yaqinlashadi** deyiladi. Agar (40.1) limit mavjud bo’lmasa, u holda (40.2) xosmas integral **mavjud emas** yoki **uzoqlashadi** deyiladi.

(-∞,b] intervalda chegaralangan *f(x)* funksiyaning xosmas integrali ham (40.2) kabi aniqlanadi:

$$\int_{-\infty}^{b} f(x)dx = \lim_{R\to-\infty}\int_R^b f(x)dx . \qquad (3)$$

Bunda *f(x)* funksiya istalgan *[R,b] (R,b)* kesmada integrallanuvchi.

Shuningdek,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx \qquad (4)$$

tenglik yordamida *f(x)* funksiyaning *(-∞,+∞)* bo’yicha xosmas integrali aniqlanadi. Bunda *c* ixtiyoriy o’zgarmas son. (4) tenglikning o’ng tomonidagi har ikkala xosmas integral ham yaqinlashgada uning chap tomonidagi xosmas interal ham yaqinlashadi.

Endi birinchi tur xosmas integralni hisoblashga misollar keltiramiz.

1- misol. $\int_0^{+\infty}\frac{dx}{1+x^2}=\lim_{R\to\infty}\int_0^R\frac{dx}{1+x^2}=\lim_{R\to\infty} arctgx\Big|_0^R=\lim_{R\to\infty} arctgR=\frac{\pi}{2},$

ya‘ni berilgan xosmas integral yaqinlashadi.

2- misol. $\int_0^{+\infty}\sin xdx=\lim_{R\to+\infty}\int_0^R\sin xdx=\lim_{R\to+\infty}(-\cos x)\Big|_0^R=\lim_{R\to+\infty}(-\cos R-1)=1-\lim_{R\to+\infty}\cos R,$

ammo *cosR* funksiya *R*→+∞ da limitga ega bo’lmaganligi uchun integral uzoqlashadi.

3-misol. $\int_1^{+\infty}\frac{dx}{x^\alpha}$, α -biror son.

Shu birinchi tur xosmas integralni *α* ning qanday qiymatlarida yaqinlashishi va qanday qiymatlarida uzoqlashishini aniqlaymiz.

a) α ≠1 bo’lsin, u holda istalgan *R>0* uchun

$$\int_1^{+\infty}\frac{dx}{x^2}\lim_{R\to+\infty}\int_1^R\frac{dx}{x^2}=\lim_{R\to+\infty}\int_1^R x^{-\alpha}dx=\lim_{R\to+\infty}\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^R=\lim_{R\to+\infty}\frac{R^{1-\alpha}-1}{1-\alpha}=\begin{cases}\frac{1}{\alpha-1}, agar\,\alpha>1, bo'lsa,\\ +\infty, agar\,\alpha<bo'lsa.\end{cases}$$

b) α=1 bo’lsin, u holda istalgan *R>0* son uchun

$$\int_1^{+\infty}\frac{dx}{x}=\lim_{R\to+\infty}\int_1^R\frac{dx}{x}=\lim_{R\to+\infty}\ln x\Big|_1^R=\lim_{R\to+\infty}\ln R=+\infty.$$

Shunday qilib, berilgan integral *α>1* bo’lganda yaqinlashadi, *α ≤1* bo’lganda esa uzoqlashadi.

Chegaralanmagan funksiyaning xosmas integrali

2-ta‘rif. *f(x)* funksiya *[a,b]* oraliqda aniqlangan bo’lsin. Agar *f(x)* funksiya *x=b* nuqtaning biror atrofida chegaralanmagan bo’lib, u *[a,b)* ga tegishli har qanday *[a,b- ε]* kesmada chegaralangan bo’lsa *x=b* nuqta *f(x)* ning maxsus nuqtasi deyiladi. *f(x)* funksiya istalgan *[a,b-ε] (ε >0,b- ε >a)* kesmada integrallashuvchi, ya‘ni $\int_a^{b-\varepsilon}f(x)dx$ aniq integral mavjud bo’lsin. U holda

$$\lim_{\varepsilon\to+0}\int_a^{b-\varepsilon}f(x)dx \qquad (40.5)$$

chekli limit mavjud bo’lsa, uni **ikkinchi tur** yoki **uzlikli funksiyaning** xos integral deb ataladi va $\int_a^b f(x)dx$ (6)

ko’rinishda belgilanadi. Bu holda (40.6) integral mavjud yoki **yaqinlashadi** deb aytiladi. (5) limit mavjud bo’lmasa, u holda (6) integral mavjud emas yoki **uzoqlashadi** deb aytiladi.

Shuningdek *x=a* nuqta *f(x)* funksiyaning maxsus nuqtasi (*a* nuqtaning yaqin atrofida *f(x)* chegaralanmagan va istalgan *[a+ε,b] (ε>0,a+ε<b)* kesmada chegaralangan) bo’lganda xosmas integral

$$\int_a^b f(x)dx = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b f(x)dx$$

kabi aniqlanadi.

Agar *f(x)* funksiya *[a,b]* kesmaning biror ichki *c* nuqtasining qandaydir atrofida chegaralanmaganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik yordamida aniqlanadi. Bu tenglikning o'ng tomonidagi har ikkala integral yaqinlashganda uning chap tomonidagi xosmas integral ham yaqinlashadi.

Shuningdek, *a* va *b* nuqtalar *f(x)* funksiyaning maxsus nuqtalari bo'lganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (7)$$

tenglik yordamida aniqlanadi, bunda *c (a,b)* intervalining ixtiyoriy nuqtasi.

Izoh. Uzluksiz funksiyaning ikkinchi tur uzilish nuqtasi uning maxsus nuqtasi bo'ladi.

4- misol. $\int_a^b \frac{dx}{(x-a)^\alpha}$ integral tekshirilsin, bunda $\alpha>0$ - biror son.

Yechish. Integral ostidagi $\frac{1}{(x-a)^\alpha}$ funksiya uchun $x=a$ maxsus nuqta bo'ladi.

a) $\alpha \neq 1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{(x-a)^\alpha} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{(x-a)^\alpha} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b (x-a)^{-\alpha}dx = \lim_{\varepsilon\to+0}\frac{(x-a)^{-\alpha+1}}{-\alpha+1}\Big|_{a+\varepsilon}^b =$$

$$= \lim_{\varepsilon\to+0}\begin{cases}\frac{(b-a)^{1-\alpha}}{1-\alpha}, agar \alpha < 1 bo'lsa,\\ +\infty, agar \alpha > 1 bo'lsa.\end{cases}$$

b) $\alpha=1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\ln|x-a|\Big|_{a+\varepsilon}^b = \lim_{\varepsilon\to+0}\left[\ln(b-a)-\ln\varepsilon\right] = +\infty.$$

Shunday qilib berilgan integral $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

Shuningdek $\int_a^b \frac{dx}{(b-x)^\alpha}$ integral ham $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

5- misol. $\int_2^5 \frac{dx}{\sqrt[3]{x-3}}$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x-3}}$ funksiya *[2,5]* kesmaning ichidagi $x=3$ nuqtada ikkinchi tur uzilishga ega, ya'ni bu nuqta funktsyaining maxsus nuqtasi.

Ta'rifga binoan:

$$\int_2^5\frac{dx}{\sqrt[3]{x-3}}=\int_2^3\frac{dx}{\sqrt[3]{x-3}}+\int_2^5\frac{dx}{\sqrt[3]{x-3}}=\lim_{\varepsilon_1\to+0}\int_2^{3-\varepsilon_1}\frac{dx}{(x-3)^{\frac{1}{3}}}+\lim_{\varepsilon_2\to+0}\int_{3+\varepsilon_2}^5\frac{dx}{(x-3)^{\frac{1}{3}}}=\lim_{\varepsilon_1\to+0}\frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\Big|_2^{3-\varepsilon_1}+\lim_{\varepsilon_2\to+0}\frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\Big|_{3+\varepsilon_2}^5=$$

$$=\lim_{\varepsilon_1\to+0}\frac{3}{2}\sqrt[3]{(x-3)^2}\Big|_2^{3-\varepsilon_1}+\lim_{\varepsilon_2\to+0}\frac{3}{2}\sqrt[3]{(x-3)^2}\Big|_2^{3-\varepsilon_2}=\frac{3}{2}\lim_{\varepsilon_1\to+0}\left[\sqrt[3]{\varepsilon_1^{\,2}}-1\right]+\frac{3}{2}\lim_{\varepsilon_2\to+0}\left[\sqrt[3]{2^2}-\sqrt[3]{\varepsilon_2^{\,2}}\right]=-\frac{3}{2}+\frac{3}{2}\sqrt[3]{4}=\frac{3}{2}\left(\sqrt[3]{4}-1\right).$$

Demak, berilgan integral yaqinlashadi.

6-misol. $\int_0^1\frac{dx}{\sqrt{x(1-x)}}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt{x(1-x)}}$ funksiya uchun *a=0, b=1* nuqtalar maxsus nuqtalar bo’ladi. Xosmas itntegralning ta‘rifidan foydalanib topamiz:

$$\int_0^1\frac{dx}{\sqrt{x(1-x)}}=\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{x(1-x)}}+\int_{\frac{1}{2}}^1\frac{dx}{\sqrt{x(1-x)}}=\lim_{\varepsilon_1\to+0}\int_{\varepsilon_1}^{\frac{1}{2}}\frac{dx}{\sqrt{x(1-x)}}+\lim_{\varepsilon_2\to+0}\int_{\frac{1}{2}}^{1-\varepsilon_2}\frac{dx}{\sqrt{x(1-x)}}=$$

$$=\lim_{\varepsilon_1\to+0}\int_{\varepsilon_1}^{\frac{1}{2}}\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}+\lim_{\varepsilon_2\to+0}\int_{\frac{1}{2}}^{1-\varepsilon_2}\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}=\lim_{\varepsilon_1\to+0}\arcsin\frac{x-\frac{1}{2}}{\frac{1}{2}}\Big|_{\varepsilon_1}^{\frac{1}{2}}+\lim_{\varepsilon_2\to+0}\arcsin\frac{x-\frac{1}{2}}{\frac{1}{2}}\Big|_{\frac{1}{2}}^{1-\varepsilon_2}=$$

$$=\lim_{\varepsilon_1\to+0}\left[\arcsin 0-\arcsin(2\varepsilon_1-1)\right]-\lim_{\varepsilon_2\to+0}\left[\arcsin(2\varepsilon_2-1)-\arcsin 0\right]=\arcsin 1+\arcsin 1=2\cdot\frac{\pi}{2}=\pi.$$

Demak integral yaqinlashuvchi va

$$\int_0^1\frac{dx}{\sqrt{x(1-x)}}=\pi\,.$$

7- misol. $\int_{-1}^1\frac{dx}{x^2}$ integral hisoblansin.

Yechish. Birinchi qarashda berilgan integral juda oson hisoblanadi, ya‘ni

$$\int_{-1}^1\frac{dx}{x^2}=-\frac{1}{x}\Big|_{-1}^1=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2\,.$$

Ammo olingan natija noto’g’ri. Bu noto’g’ri natija e‘tiborsizligimiz oqibatida, ya‘ni integral ostidagi $\frac{1}{x^2}$ funksiya *[-1,1]* kesmada *x=0* maxsus nuqtaga ega ekanligini hisobga olmaganligimiz sababli keldik. Biz berilgan integralni oddiy aniq integral deb emas, balki xosmas integral deb qarashimiz lozim. Uni ta‘rifdan foydalanib hisoblaymiz:

$$\int_{-1}^1\frac{dx}{x^2}=\int_{-1}^0\frac{dx}{x^2}+\int_0^1\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}+\lim_{\varepsilon_2\to+0}\int_{\varepsilon_2}^1\frac{dx}{x^2}. \qquad (8)$$

Limitlardan birini hisoblaymiz: $\lim_{\varepsilon_1\to+0}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\left(-\frac{1}{x}\right)\Big|_{-1}^{\varepsilon}=\lim_{\varepsilon\to+0}\left(+\frac{1}{\varepsilon}--\frac{1}{-1}\right)=\infty.$

(40.8) tenglikning o’ng tomonidagi xosmas integrallardan biri uzoqlashganligi uchun ta‘rifga binoan uning chap tomonidagi xosmas integral ham uzoqlashadi.

40.4. Xosmas integrallarning yaqinlashish alomatlari

Ko'p hollarda xosmas integralning qiymatini topish talab etilmasdan uning yaqinlashuvchi yoki uzoqlashuvchi ekanini bilishning o'zi kifoya qiladi. Bunday hollarda **taqqoslash teoremalari** deb ataluvchi teoremadan foydalanish mumkin.

1. teorema. Agar *f(x)* va *φ (x)* funksiyalar *[a, + ∞)* oraliqda uzluksiz bo'lib, $0 \le f(x) \le \varphi(x)$ shartni qanoatlantirsa, u holda

a) $\int_a^{+\infty} \varphi(x)dx$ xosmas integral yaqinlashsa,

$\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

b) $\int_a^{+\infty} f(x)dx$ integrla uzoqlashganda $\int_a^{+\infty} \varphi(x)dx$ integral ham uzoqlashadi.

Bu teorema faqatgina nomanfiy funksiyalarga tegishli bo'lib undan ishorasini saqlamaydigan funksiyalarning xosmas integrallarini tekshirishda foydalanib bo'lmaydi. Bunday holda quyidagi teoremadan foydalanish mumkin.

2.-teorema. $\int_a^{+\infty} |f(x)|dx$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

Bunda oxirgi integral **absolyut yaqinlashuvchi** deyiladi.

$\int_a^{+\infty} f(x)dx$ yaqinlashuvchi $\int_a^{+\infty} |f(x)|dx$ integral uzoqlashuvchi bo'lganda $\int_a^{+\infty} f(x)dx$ integral **shartli yaqinlashuvchi** deyiladi.

8- misol. $\int_1^{+\infty} \frac{dx}{x^3(1+e^x)}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{x^3(1+e^x)}$ funksiyani $\frac{1}{x^3}$ funksiya bilan taqqoslaymiz. Barcha *x≥1* uchun $\frac{1}{x^3(1+e^x)} \le \frac{1}{x^3}$ bo'lib, $\int_1^{+\infty} \frac{dx}{x^3}$ yaqinlashganligi (3-misolga qarang) uchun.1. teoremaning *a)* bandiga binoan berilgan integral ham yaqinlashadi.

9- misol. $\int_1^{+\infty} \frac{\sqrt{x}}{1+x}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{\sqrt{x}}{1+x}$ funksiyani $\frac{1}{2\sqrt{x}}$ funksiya bilan taqqoslab barcha

x≥1 uchun $\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x} = \frac{\sqrt{x}}{x+x} \le \frac{\sqrt{x}}{1+x}$ ga ega bo'lamiz. $\alpha = \frac{1}{2} < 1$ bo'lgani uchun $\int_1^{+\infty} \frac{dx}{2\sqrt{x}}$ integral uzoqlashadi (3- misolga qarang).

1. teoremaning *b)* qismiga ko'ra berilgan integral ham uzoqlashadi.

10-misol. $\int_{1}^{+\infty}\frac{\sin x}{x^4}dx$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\sin x}{x^4}$ funksiya $[1,+\infty)$ da ishorasini saqlamaydi. Shuning uchun $\int_{1}^{+\infty}\left|\frac{\sin x}{x^4}\right|dx$ integralni qaraymiz. $[1,+\infty)$ da $0\le\left|\frac{\sin x}{x^4}\right|=\frac{|\sin x|}{x^4}\le\frac{1}{x^4}$ bajarilib $\int_{1}^{+\infty}\frac{dx}{x^4}$ (3-misolga qarang) yaqinlashganligi uchun 40.1-teoremaning *a)* bandiga ko'ra $\int_{1}^{+\infty}\left|\frac{\sin x}{x^4}\right|dx$ yaqinlashadi.

40.2-teoremaga ko'ra berilgan integral ham yaqinlashadi. U absolyut yaqinlashadi.

Ikkinchi tur xosmas integral uchun ham 40.1 va 40.2 teoremaga o'xshash teoremalar mavjud.

1´-teorema. *f(x)* va *φ(x)* funksiyalar *(a,b]* oraliqda uzluksiz bo'lib *x=a* nuqta ularning ikkinchi tur uzilish nuqtasi (maxsus nuqtasi) bo'lsin. Agar *(a,b]* oraliqning barcha nuqtalarida

$$0\le f(x)\le \varphi(x)$$

tengsizlik bajarilsa, u holda: *a)* $\int_{a}^{b}\varphi(x)dx$ xosmas integral yaqinlashsa $\int_{a}^{b}f(x)dx$ xosmas integral ham yaqinlashadi.

b) $\int_{a}^{b}f(x)dx$ xosmas integral uzoqlashsa $\int_{a}^{b}\varphi(x)dx$ xosmas integral ham uzoqlashadi.

2´-teorema. *a=x* nuqta uzluksiz *f(x)* funksiyaning maxsus nuqtasi bo'lib $\int_{a}^{b}|f(x)|dx$ xosmas integral yaqinlashsa $\int_{a}^{b}f(x)dx$ xosmas integral ham yaqinlashadi.

11-misol. $\int_{0}^{1}\frac{dx}{\sqrt[3]{x+3x^3}}$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x+3x^2}}$ funksiya *(0,1]* oraliqda uzluksiz bo'lib u *x=0* nuqtada ikkinchi tur uzilishga ega. *(0,1]* oraliqdagi barcha *x* lar uchun $\frac{1}{\sqrt[3]{x+3x^2}}<\frac{1}{\sqrt[3]{x}}$

tengsizlik bajarilib $\int_{0}^{1}\frac{dx}{\sqrt[3]{x}}$ xosmas integral yaqinlashgani uchun (4-misolga qarang). 40.1´-teoremaning *a)* bandiga binoan qaralayotgan xosmas integral ham yaqinlashadi.

12-misol. $\int_{1}^{2}\frac{2+\sin x}{(x-1)^2}dx$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{2+\sin x}{(x-1)^2}$ funksiya *(1,2]* oraliqda uzluksiz bo'lib u *x=1* nuqtada ikkinchi tur uzilishga ega, uning surati *x* ning istalgan qiymatida *2+sinx≥1,* chunki *sinx*≥-1.

Shuning uchun. $\frac{2+\sin x}{(x-1)^2} \geq \frac{1}{(x-1)^2}$. Biroq $\int_1^2 \frac{dx}{(x-1)^2}$ uzoqlashadi, chunki *α=2>1* (4-misol). Demak 40.1′ teoremaning *b)* bandiga ko'ra berilgan integral ham uzoqlashadi.

13-misol. $\int_0^{\pi} \frac{\cos x dx}{\sqrt{x}}$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\cos x dx}{\sqrt{x}}$ funksiya *x=0* maxsus nuqtaga ega. *cosx* funksiya [0,π] kesmada ishorasini saqlamaydi, ya'ni u $[0,\frac{\pi}{2}]$ da musbat, $[\frac{\pi}{2},\pi]$ da manfiy. Shuning uchun $\int_0^{\pi} \left|\frac{\cos x}{\sqrt{x}}\right| dx$**.** Integralni qaraymiz. Barcha *x* lar uchun $|\cos x| \leq 1$ ekanini hisobga olsak [0,π] oraliqdagi barcha *x* lar uchun $\left|\frac{\cos x}{\sqrt{x}}\right| \leq \frac{1}{\sqrt{x}}$ ekanligi kelib chiqadi.

$\int_0^{\pi} \frac{dx}{\sqrt{x}}$ xosmas integral yaqinlashganligi (4-misol) uchun 40.1′-teoremaning *a)* bandiga binoan $\int_0^{\pi} \left|\frac{\cos x}{\sqrt{x}}\right| dx$ xosmas integral ham yaqinlashadi.

40.2′-teoremaga ko'ra berilgan xosmas integral ham yaqinlashadi. Demak u absolyut yaqinlashadi.

Mustaqil yechish uchun mashqlar.

Quyidagi xosmas integrallarni hisoblang.

1. $\int_1^{+\infty} \frac{dx}{x^5}$. Javob: $\frac{1}{4}$.
2. $\int_0^{+\infty} \frac{dx}{4+x^2}$ Javob: $\frac{\pi}{4}$.
3. $\int_{\infty}^{+\infty} \frac{dx}{x^2+2x+2}$ Javob: π.
4. $\int_0^{1} \frac{dx}{\sqrt[3]{x}}$ Javob: $\frac{3}{2}$.
5. $\int_0^{2} \frac{dx}{\sqrt{4-x^2}}$ Javob: $\frac{\pi}{2}$.

6. $\int_{4}^{8} \frac{dx}{\sqrt[3]{x-5}}$ Javob: $\frac{3}{2}(\sqrt[3]{9}-1)$.

Quyidagi integrallarning yaqinlashishi yoki uzoqlashishini tekshiring.

7. $\int_{4}^{+\infty} \frac{x^2 dx}{x^3+1}$. Javob: uzoqlashadi.

8. $\int_{1}^{+\infty} \frac{xdx}{\sqrt[3]{(1+x^2)^2}}$ Javob: uzoqlashadi.

9. $\int_{-2}^{+\infty} \frac{dx}{x^2+\sqrt{x^4+3}}$ Javob: yaqinlashadi.

10. $\int_{0}^{\frac{\pi}{2}} tgxdx$. Javob: uzoqlashadi.

11. $\int_{0}^{5} \frac{dx}{(x-2)^2}$. Javob: uzoqlashadi.

12. $\int_{0}^{5} \frac{dx}{x^2-6x+8}$. Javob: uzoqlashadi.

13. $\int_{2}^{6} \frac{2+\cos x}{(x-2)^3} dx$. Javob: uzoqlashadi.

41-ma‘ruza. Mavzu: Aniq integrallarni taqribiy hisoblash

Reja:

1. Masalaning qo’yilishi.
2. To’g’ri to’rtburchaklar formulasi.
3. Trapetsiyalar formulasi.
4. Simpson formulasi.

Adabiyotlar: 1,4,7,9,10.

Tayanch iboralar: taqribiy hisoblash, absolyut va nisbiy xato, trapetsiya, parabola, egri chiziqli trapetsiya.

41. 1. Masalaning qo’yilishi

$\int_a^в f(x)dx$ aniq integralni hisoblash talab etilsin, bunda *f(x)* *[a,b]* kesmada uzluksiz funksiya. Agar integral ostidagi *f(x)* funksiyaning *F(x)* boshlang’ich funksiyasini topish imkoni bo’lsa, u holda berilgan integral

$$\int_a^в f(x)dx = F(b) - F(a)$$

Nyuton-Leybnis formulasi yordamida hisoblanadi.

Biroq xatto uzluksiz funksiyaning boshlang’ich funksiyasi ham har doim elementar funksiya bo’lavermasligini ko’rdik. Masalan,

$$\frac{\sin x}{x}, \sin x^2, \frac{1}{\ell nx}, e^{-x^2}$$

ko’rinishdagi va boshqa ko’pgina uzluksiz funksiyalarning boshlang’ich funksiyalari mavjud bo’lsada, ular elementar funksiya orqali ifodalanmaydi. Bunday hollarda aniq integralning aniq qiymatini Nyuton-Leybnis formulasidan foydalanib topishning iloji bo’lmaydi. Uni istalgan aniqlikda taqribiy hisoblash mumkin. Aniq integralni taqribiy hisoblash usuli ko’p hollarda shu integralning geometrik ma‘nosiga ya‘ni $\int_a^b f(x)dx$ aniq integral son qiymat jihatdan yuqoridan uzluksiz *y= f(x)* *(f(x)≥0)* egri chiziq, quyidan *0x* o’qning *[a,b]* kesmasi va yon tomonlardan *x=a*, *x=b* vertikal to’g’ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuziga tengligiga asoslanadi. Shunga ko’ra aniq integralni taqribiy hisoblash masalasi egri chiziqli trapetsiyaning yuzini taqribiy hisoblashga keladi.

Aniq integralni taqribiy hisoblash g’oyasi shundan iboratki bunda *y= f(x)* egri chiziq o’ziga «yaqin» yangi egri chiziqqa almashtiriladi. Natijada izlanayotgan yuza taqriban egri chiziq bilan chegaralangan egri chiziqli trapetsiyaning yuziga teng bo’ladi.

Yangi egri chiziq sifatida odatda egri chiziqli trapetsiyaning yuzasi osonlikcha hisoblanadigan egri chiziq tanlanadi. Shu egri chiziqqa bog’liq ravishda u yoki bu taqribiy integrallash formulalariga ega bo’lamiz.

41.2. To'g'ri to'rtburchaklar formulasi

Faraz qilaylik $y= f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lib

$$\int_a^в f(x)dx$$

aniq integralni hisoblash talab etilsin.

$[a,b]$ kesmani $a=x_o, x_1, x_2,\dots x_к,\dots, x_n=b$ nuqtalar bilan n ta teng qismga ajratamiz.

Har bir bo'lakning uzunligi

$$\Delta x = \frac{b-a}{n}$$

bo'lishi ravshan.

$f(x)$ funksiyaning $x_o, x_1, x_2,\dots, x_к,\dots, x_n$ nuqtalardagi qiymatlari $y_o= f(x_o), y_1= f(x_1), y_2= f(x_2),\dots, y_к= f(x_к),\dots, y_n= f(x_n)$ larni hisoblaymiz.

$$y_o\Delta x+y_1\Delta x+y_2\Delta x+\dots+y_{n-1}\Delta x=\sum_{к=0}^{n-1} y_k\Delta x \text{ va}$$

$$y_1\Delta x+ y_2\Delta x+ y_3\Delta x+\dots+ y_n\Delta x =\sum_{к=1}^{n} y_k\Delta x$$

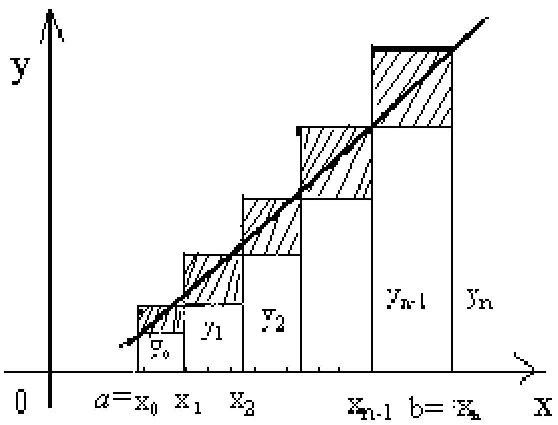
yig'indilarni tuzamiz.

Bu yig'indilarning har biri $[a,b]$ kesmada uzluksiz $f(x)$ funksiyaning integral yig'indisi bo'ladi va shuning uchun ular taqriban integralni ifodalaydilar:

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_0+y_1+\dots+y_{n-1}) = \frac{b-a}{n}\sum_{k=0}^{n-1} y_k\,, \qquad (41.1)$$

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_1+y_2+\dots+y_n) = \frac{b-a}{n}\sum_{k=1}^{n} y_k\,. \qquad (41.2)$$

Aniq integralni taqribiy hisoblash uchun chiqargan bu formulalarimiz to'g'ri to'rtburchak formulasi deb yuritiladi.



142-chizma.

142-chizmadan ko'rinib turibdiki, agar $f(x)$ funksiya musbat va $[a,b]$ kesmada o'suvchi bo'lsa, (41.1) formula berilgan integralning taqribiy qiymatini kami bilan, (41.2) formula esa ortig'i bilan ifodalaydi.

Shu chizmada bu ikki (41.2) va (41.1) yig'indi orasidagi ayirma shtrixlangan n ta to'g'ri to'rtburchak yuzlarining yig'indisiga teng ekanligi ko'rsatilgan:

$$\delta_n = \frac{b-a}{n}[f(b) - f(a)].$$

Binobarin, $[a,b]$ kesmada o'suvchi bo'lgan $f(x)$ funksiyadan olingan integralni hisoblashda yo'l qo'yilgan xato bu δ_n ayirmadan katta bo'lmaydi. Bo'linishlar soni n ni orttira borib, bu ayirmani istalgancha kichik qilib olish, demak, $\int_a^b f(x)dx$ integralni oldindan berilgan istalgan aniqlik bilan hisoblash mumkin.

Agar $f(x)\geq 0$ funksiya $[a,b]$ kesmada kamaysa

$$\delta_n = \frac{b-a}{n}[f(a) - f(b)]$$

bo'ladi.

Agar $f(x)$ funksiya $[a,b]$ da monoton bo'lmasa, $f''(x)$ hosila esa mavjud bo'lsa va bu kesmada chegaralangan bo'lsa (41.1) va (41.2) formulalarning δ_n xatoligi uchun ushbu baho o'rinlidir:

$$|\delta_n| \leq \frac{M_1(b-a)^2}{2n},$$

bu yerda M_1- $f''(x)$ ning $[a,b]$ kesmadagi moduli maksimumi.

1-misol. $\int_1^2 \sqrt[3]{x}dx$ integral integrallash oralig'ini $n=8$ bo'lakka bo'lib to'g'ri to'rtburchaklar formulalari yordamida taqribiy hisoblansin. Integral aniq hisoblanib natijaning **absolyut** hamda **nisbiy** xatolari topilsin.

Yechish. Quyidagiga egamiz: $y = \sqrt[3]{x}, n = 8, \Delta x = \frac{2-1}{8} = 0{,}125.$

Ushbu jadvalni tuzamiz:

I	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
x_i	*1*	*1,125*	*1,250*	*1,375*	*1,500*	*1,625*	*1,750*	*1,875*	*2*
y_i	*1*	*1,042*	*1,073*	*1,122*	*1,145*	*1,117*	*1,205*	*1,233*	*1,260*

(41.1) formulaga ko'ra

$$\int_1^2 \sqrt[3]{x}dx \simeq 0{,}125(y_o + y_1 + \ldots + y_7) = 0{,}125\cdot 8{,}987 = 1{,}123375,$$

(41.2) formulaga binoan

$$\int_1^2 \sqrt[3]{x}dx \simeq 0{,}125(y_1 + y_2 + \ldots, + y_8) = 0{,}125\cdot 9{,}247 = 1{,}155875$$

taqribiy tengliklarga ega bo'lamiz.

Endi berilgan integralni aniq qiymatini Nyuton-Leybnis formulasidan foydalanib topamiz:

$$\int_1^2 \sqrt[3]{x}dx = \int_1^2 x^{\frac{1}{3}}dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Big|_1^2 = \frac{3}{4}\sqrt[3]{x^4}\Big|_1^2 = \frac{3}{4}(\sqrt[3]{2^4}-1) \simeq 1{,}14.$$

Absolyut va nisbiy xatolar mos ravishda (41.1) formula uchun

$|1{,}14-1{,}123375|=0{,}016625 \simeq 0{,}0167$ va $\frac{0{,}0167}{1{,}14}\cdot 100\% = 1{,}5\%,$ (41.2) formula uchun

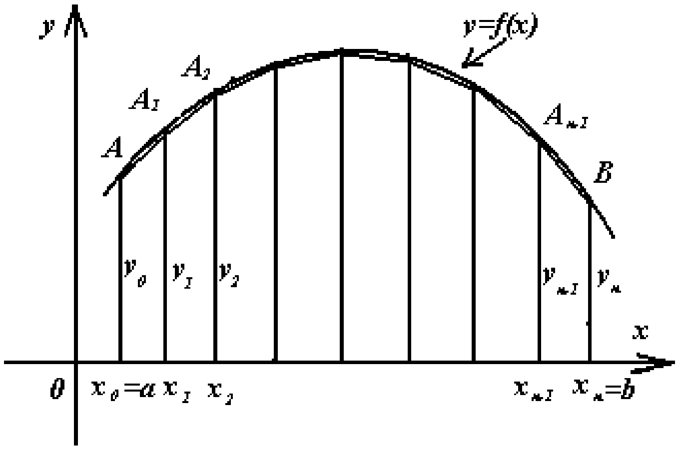
$|1{,}14-1{,}155875|=0{,}015875 \simeq 0{,}0158$ va $\frac{0{,}0158}{1{,}14}\cdot 100\% \simeq 1{,}4\%$ bo'ladi.

41.2. Trapetsiyalar formulasi

[a,b] kesmada uzluksiz *f(x)* funksiya berilgan bo'lib $\int_a^b f(x)dx$ aniq integralni hisoblash talab etilsin.

[a,b] kesmani $a= x_o, x_1, x_2,..., x_к,..., x_n=b$ nuqtalar bilan uzunligi $\Delta x = \frac{b-a}{n}$ bo'lgan *n* ta teng bo'laklarga ajratamiz.

So'ngra $y= f(x)$ funksiyaning $x_o, x_1, x_2,..., x_к,..., x_n$ nuqtalardagi qiymatlari $y_o= f(x_o), y_1= f(x_1), y_2= f(x_2),..., y_к= f(x_к),..., y_n= f(x_n)$ larni hisoblaymiz.



143-chizma.

$\int_a^b f(x)dx$ aniq integral *aABb* egri chiziqli trapetsiya yuzini ifodalar edi. $y= f(x)$ egri chiziqni unga ichki chizilgan siniq chiziq bilan almashtiramiz. Bu holda *aABb* egri chiziqli trapetsiyaning yuzi yuqoridan AA_1, A_1A_2,... $A_{n-1}B$ vatarlar bilan chegaralangan oddiy trapetsiyalar yuzlarining yig'indisiga teng bo'ladi.

Trapetsiyaning yuzi asoslari yig'indisining yarmi bilan balandligi ko'paytmasiga teng bo'lganligi sababli 1-trapetsiyaning yuzi $\frac{y_0+y_1}{2}\Delta x,$ 2-sining yuzi $\frac{y_1+y_2}{2}\Delta x$ va hokazo oxirgisining yuzi $,\frac{y_{n-1}+y_n}{2}\Delta x$ ga teng bo'ladi. Shuning uchun

$$\int_a^b f(x)dx \approx \left(\frac{y_0+y_1}{2}\Delta x + \frac{y_1+y_2}{2}\Delta x + \ldots + \frac{y_{n-1}+y_n}{2}\Delta x\right) \text{ yoki}$$

$$\int_a^в f(x)dx \approx \frac{в-a}{n}\left(\frac{y_0+y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right). \qquad (41.3)$$

(41.3) trapetsiyalar formulasi deb ataladi. n son qancha katta bo'lsa (41.3) taqribiy tenglikning o'ng tomonidagi yig'indi shuncha katta aniqlik bilan berilgan integralning qiymatini beradi.

(41.3) formulaning xatoligi $|\delta_n| \le \frac{M_2(b-a)^3}{12n^2}$ dan oshmaydi,

bu yerda M_2- $f''(x)$ funksiyaning $[a,b]$ kesmadagi moduli maxsimumi

2-misol. $\int_0^{1,6} \sin(x^2)dx$ aniq integral $[0; 1,6]$ kesmani $n=8$ ta teng bo'lakka bo'lib trapetsiyalar formulasi yordamida taqribiy hisoblansin.

Yechish. Integral ostidagi funksiyaning $n=8$ va $\Delta x = \frac{b-a}{n} = \frac{1,6-0}{8} = 0,2$ bo'lgandagi qiymatlari jadvalini tuzamiz.

	x_i	x_i^2	$y_i=\sin(x_i^2)$
	0	0	0,0000
	0,2	0,04	0,0400
	0,4	0,16	0,1593
	0,6	0,36	0,3523
	0,8	0,64	0,5972
	1,0	1	0,8415
	1,2	1,44	0,9915
	1,4	1,96	0,9249
	1,6	2,56	0,5487

$n=8$ bo'lganda (41.3) formulaga binoan

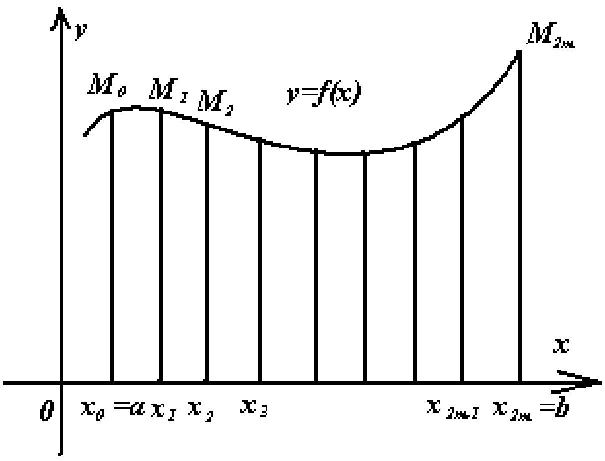
$$\int_0^{1,6} \sin(x^2)dx \approx \Delta x\left(\frac{y_0+y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right) = 0,2\left(\frac{0+0,5487}{2} + 0,0400 +\right.$$

$$\left. + 0,1593 + 0,3523 + 0,5972 + 0,8415 + 0,9915 + 0,9249\right) = 0,2 \cdot 4,1807 = 0,8362.$$

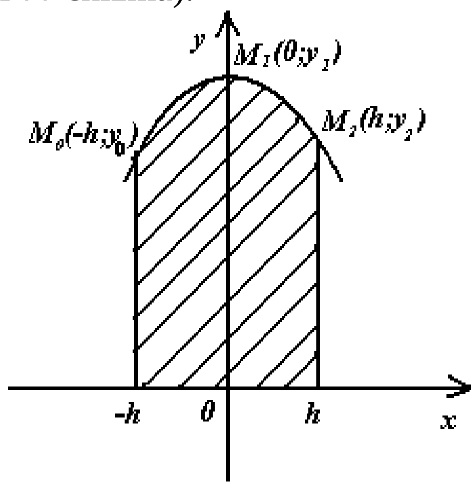
ga ega bo'lamiz.

41.4. Simpson formulasi

$\int_a^b f(x)dx$ integralni hisoblash talab etilsin, bunda *f(x)* *[a,b]* kesmada uzluksiz funksiya. *[a,b]* kesmani juft sondagi *n=2m* ta teng bo'laklarga bo'lamiz. Birinchi ikkita $[x_o, x_1]$, $[x_1, x_2]$ oraliqlarga mos keluvchi egri chiziqli trapetsiyaning yuzini simmetriya o'qi *0y* o'qga parallel bo'lib $M_0(x_o, y_o)$, $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ nuqtalardan o'tuvchi parabola bilan chegaralangan egri chiziqli trapetsiya yuzi bilan almashtiramiz. Yuqorida aytilgan parabolaning umumiy tenglamasi $y=Ax^2+Bx+C$ ko'rinishda bo'ladi. *A,B,C* o'zgarmas sonlar parabolaning berilgan uchta nuqtadan o'tish shartidan aniqlanadi. Qolgan har juft kesmalar uchun ham shunga o'xshash parabolalar yasaymiz. Yuqoridan parabolalar bilan chegaralangan parabolik trapetsiyalar yuzlarining yig'indisi *aABb* egri chiziqli trapetsiya yuzini ya'ni aniq integral qiymatini taqriban ifodalaydi. (144-chizma).



144-chizma.



145-chizma.

Lemma. Agar egri chiziqli trapetsiya $y=Ax^2+Bx+C$ parabola, *0x* o'q hamda orasidagi masofa *2h* ga teng bo'lgan y_o, y_2 ordinatalar bilan chegaralangan bo'lsa uning yuzi

$$s = \frac{h}{3}(y_0 + 4y_1 + y_2) \qquad (41.4)$$

formula orqali topiladi, bunda y_1-kesmaning o'rtasiga mos keluvchi egri chiziq ordinatasi.

Isboti. Yordamchi koordinatalar sistemasini 145-chizmada ko'rsatilganidek qilib joylashtiramiz.

Shartga binoan $y=Ax^2+Bx+C$ parabola M_0, M_1, M_2 nuqtalardan o'tganliklari sababli bu nuqtalarning koordinatalari parabola tenglamasini qanoatlantiradi:

$$\left.\begin{array}{lll} x_0 = -h & bo'lsa & y_0 = Ah^2 - Bh + C, \\ x_1 = 0 & bo'lsa & y_1 = \qquad\quad C, \\ x_2 = h & bo'lsa & y_2 = Ah^2 - Bh + C. \end{array}\right\} \quad (41.5)$$

Qaralayotgan parabolik trapetsiyaning yuzini aniq integral yordamida aniqlaymiz.

$$S=\int_{-h}^{h}(Ax^2+Bx+C)dx=\left[\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right]\Big|_{-h}^{h}=\left(\frac{Ah^3}{3}+\frac{Bh^2}{2}+Ch\right)-$$
$$-\frac{A(-h)^3}{3}-\frac{B(-h)^2}{2}-C(-h)=\frac{2}{3}Ah^3+2Ch=\frac{h}{3}(2Ah^2+6C).$$

Ikkinchi tomondan sistemaning ikkinchi tenglamasini *4* ga ko'paytirib barcha tenglamalarni qo'shsak

$y_o+4y_1+y_2=2A\,h^2+C$ bo'ladi.

Demak, $s=\frac{h}{3}(\;y_o+4y_1+y_2)$. Lemma isbot bo'ldi. (41.4) formuladan foydalanib quyidagi taqribiy tengliklarga ega bo'lamiz. *(h=Δx).*

$$\int_{a=x_0}^{x_2}f(x)dx\approx\frac{\Delta x}{3}(y_0+4y_1+y_2),$$

$$\int_{x_2}^{x_4}f(x)dx\approx\frac{\Delta x}{3}(y_2+4y_3+y_4),$$

....

$$\int_{x_{2m-2}}^{x_{2m}}f(x)dx\approx\frac{\Delta x}{3}(y_{2m-2}+4y_{2m-1}+y_{2m}).$$

Bu tengliklarni chap va o'ng tomonlarni mos ravishda qo'shsak

$$\int_a^b f(x)dx\approx\frac{\Delta x}{3}\ (y_o+4y_1+y_2+y_2+4y_3+y_4+\ldots+y_{2m-2}+4y_{2m-1}+y_{2m})$$

yoki

$$\int_a^b f(x)dx\approx\frac{b-a}{6m}\ [y_o+y_{2m}+2(y_2+y_4+\ldots+y_{2m-2})+4(y_1+y_3+\ldots+y_{2m-1})] \quad (41.6)$$

hosil bo'ladi. Bu formula **Simpson** yoki **parabolalar** formulasi deb ataladi.

Aniq integralni Simpson formulasi yordamida taqribiy hisoblaganda xatolik

$$\frac{M_4(b-a)^5}{180(2m)^4}$$

dan oshmaydi, bu yerda $M_4-f^{IV}(x)$ funksiyaning *[a,b]* kesmadagi moduli maksimumi.

3-misol. $\int_0^{1,6}\sin(x^2)dx$ integral integrallash oralig'ini *2m=8* bo'lakka bo'lib Simpson formulasi yordamida taqribiy hisoblansin.

Yechish. *2m=8,* $\Delta x=\frac{b-a}{2m}=\frac{1,6-0}{8}=0,2$ bo'lganda (41.6) formulaga asosan $\int_0^{1,6}\sin x^2dx\approx\frac{1,6-0}{24}[y_o+y_8+2(y_2+y_4+y_6)+4y_1+y_3+y_5+y_7)]$

taqribiy tenlikka ega bo'lamiz.

2-misolda *sin(x²)* funksiyaning qiymatlari uchun tuzilgan jadvaldan foydalansak

$$\int_0^{1,6}\sin(x^2)dx \approx \frac{1,6-0}{24}[0+0,5487+2(0,1592+0,5972+0,9915)+$$
$$+4(0,0400+0,3523+0,8415+0,9249)=0,8455.$$

kelib chiqadi .

Shunday qilib $\int_0^{1,6}\sin(x^2)dx$ integralni integrallash oralig'ini *8* ga bo'lib trapetsiyalar formulasi yordamitda taqribiy hisoblaganimizda uning qiymati *0,8362* ga, Simpson formulasi yordamida hisoblaganda *0,8455* ga tengligini ko'rdik.

Shu integralning *0,00001* aniqlikda hisoblangan jadval qiymati *0,84528* ga tengdir.

Demak, Simpson formulasi trapetsiyalar formulasiga nisbatan qaralayotgan integralning aniqroq qiymatini berar ekan.

O'z-o'zini tekshirish uchun savollar

1. Nima uchun aniq integral taqribiy hisoblanadi?
2. Aniq integralni taqribiy hisoblash g'oyasi nimaga asoslangan?
3. To'g'ri to'rtburchaklar formulasini yozing.
4. Trapetsiyalar formulasini yozing.
5. Simpson formulasini yozing.
6. To'g'ri to'rtburchaklar formulasida xato qanday baholanadi?
7. Trapetsiyalar formulasida xato qanday baholanadi?
8. Simpson formulasida xato qanday baholanadi?
9. Keltirilgan taqribiy hisoblash formulalaridan qaysi biri aniqroq qiymatni beradi?
10. Xatoni kamaytirish uchun nima qilish kerak?

Mustaqil yechish uchun mashqlar

1. $\int_0^1 \frac{dx}{1+x^2}$ intgralni *0,06* dan katta bo'lmagan xatolik bilan to'g'ri to'rtburchaklar formulasidan foydalanib hisoblang.

Javob: *0,80998*(ortig'i bilan), *0,75998* (kami bilan).

2. $\int_0^1 \frac{dx}{1+x^2}$ intgralni trapetsiyalar formulasi bo'yicha *n=10* deb hisoblang. Javob: *0,78498*.

3. $\int_0^1 \frac{dx}{1+x^2}$ integralni *n=2m=4* deb Simpson formulasi bo'yicha taqribiy hisoblang. Javob: *0,7854*.

<u>42-ma‘ruza.</u> Mavzu: Aniq integral yordamida yuzni hisoblash

Reja:

1. Dekart koordinatalar sistemasida yuzlarni hisoblash.
2. Qutb koordinatalar sistemasida egri chiziqli sektorning yuzi.

Adabiyotlar: 1,4,7,9,10,12.

Tayanch iboralar: Dekart, qutb sistemalari, egri chiziqli trapetsiya, egri chiziqli sektor, ellips, kardioida, uch yaproqli gul, to'rt yaproqli gul, astroida, Dekart yaprog'i sirtmogi.

42.1. Dekart koordinatalar sistemasida yuzlarni hisoblash

Agar uzluksiz *f(x)* funksiya *[a,b]* kesmada nomanfiy bo'lsa, u holda y=*f(x)* egri chiziq, *ox* o'q, *x=a* va *x=b* vertikal to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzi

$$Q=\int_a^b f(x)dx \qquad (42.1)$$

ga teng bo'lishini ko'rgan edik (aniq integralning geometrik ma‘nosi) (146-chizma).

Agar *[a,b]* kesmada $f(x)\le 0$ bo'lsa, u holda aniq integralning 4-xossasiga binoan
$\int f(x)dx\le 0$ bo'ladi. Bu holda tegishli egri chiziqli trapetsiyaning yuzi

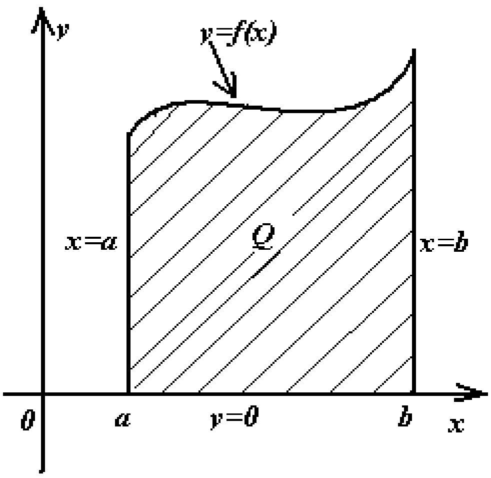
$$-\int_a^b f(x)dx=\int_a^b |f(x)|\,dx$$

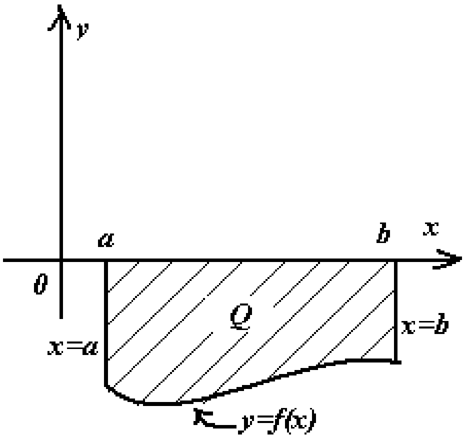
ga teng bo'ladi (147-chizma).

Shuning uchun *f(x)* funksiya *[a,b]* kesmada ishorasini o'zgartganda tegishli egri chiziqli trapetsiyaning yuzi

$$Q=\int_a^b |f(x)|\,dx \qquad (42.2)$$

ga teng bo'ladi.



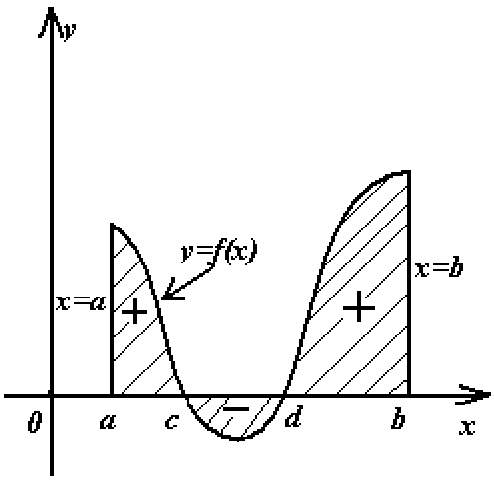


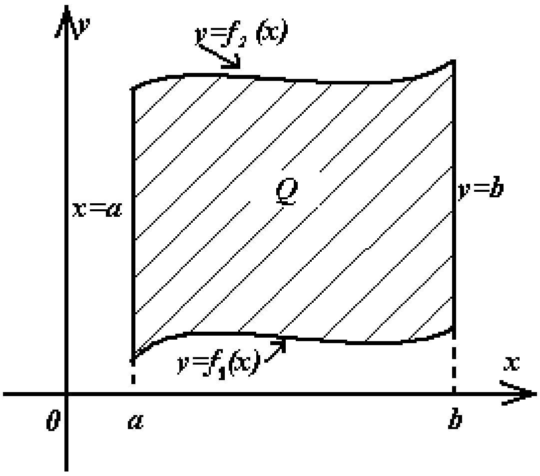
146-chizma. 147-chizma.

Masalan, 148-chizmada tasvirlangan yuzni quyidagicha topish mumkin:

$$Q=\int_a^c f(x)dx-\int_c^d f(x)dx+\int_d^b f(x)dx.$$

Bu holda (42.1) formuladan foydalanilsa *0x* o'qning yuqorisida joylashgan figuraning yuzi bilan uning quyisida joylashgan figuraning yuzini ayirmasi topiladi.





148-chizma. 149-chizma.

Bizga yuzlarni ayirmasini emas balki ularni yig'indisini topish so'ralganligi sababli *0x* o'qning pastida joylashgan figuraning yuzini topish uchun integral oldida minus ishora olinadi.

Yuqoridan uzluksiz $y=f_2(x)$, quyidan uzluksiz $y=f_1(x)$, egri chiziqlar bilan va yon tomonlardan $x=a$, $x=b$ $(a<b)$ vertikal to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzini topish uchun

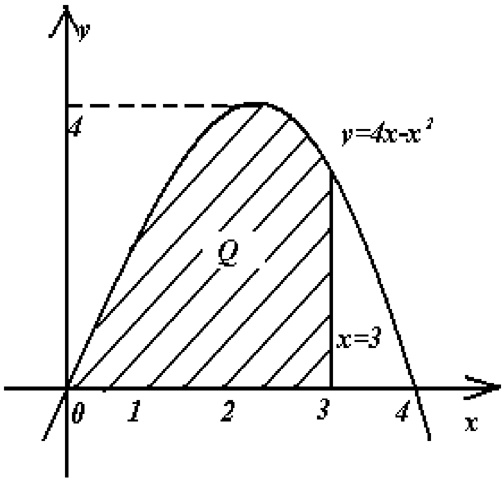
$$Q=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx$$

yoki

$$Q=\int_a^b [f_2(x)-f_1(x)]dx \qquad (42.3)$$

formulaga ega bo'lamiz(149-chizma).

1-misol. $y=4x-x^2$, $x=3$, $y=0$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang. (150-chizma).

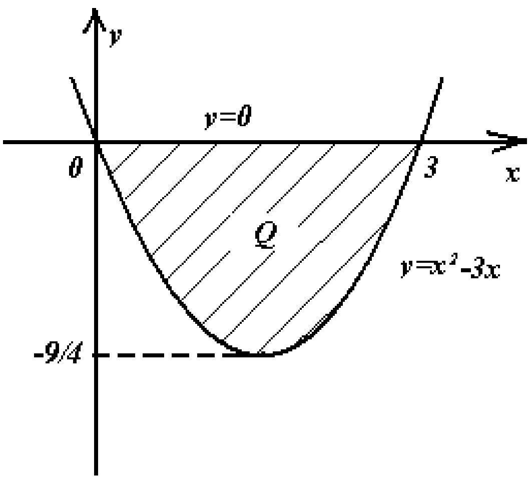


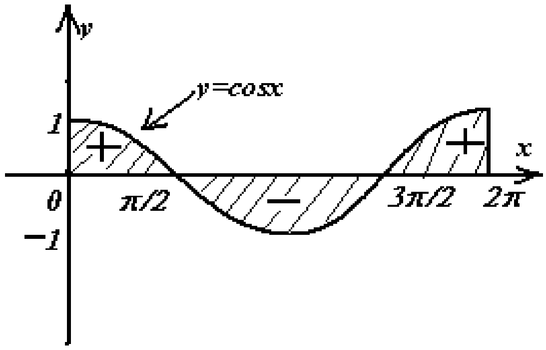
150-chizma.

Yechish. (42.1) formuladan foydalanib topamiz.

$$Q=\int_0^3(4x-x^2)dx=\left(4\cdot\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^3=2\cdot 3^2-\frac{3^3}{3}=18-9=18.$$

2-misol. $y=x^2-3x$, $y=0$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang (151-chizma).





151-chizma. 152-chizma.

Yechish. (42.2) formulaga binoan topamiz:

$$Q=-\int_0^3(x^2-3x)dx=-\left(\frac{x^3}{3}-3\cdot\frac{x^2}{2}\right)\Big|_0^3=-\left(\frac{3^3}{3}-3\cdot\frac{3^2}{2}\right)=-(9-13{,}5)=4{,}5.$$

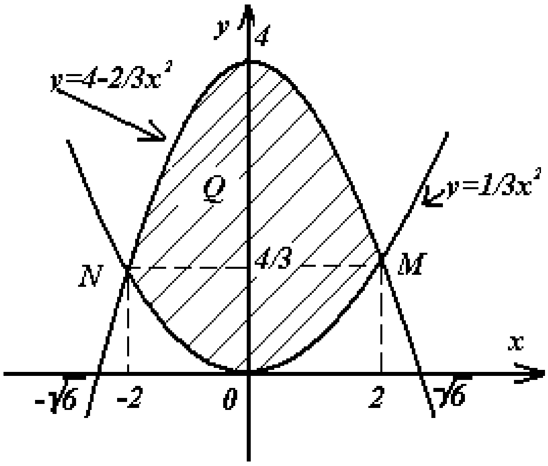
3-misol. $0\le x\le 2\pi$ bo'lganda $y=\cos x$ kosinusoida va $0x$ o'q bilan chegaralangan figuraning yuzi topilsin (152-chizma).

Yechish. $\left[0,\ \frac{\pi}{2}\right]$ da $cosx \le 0$, $\left[\frac{\pi}{2},\ \frac{3\pi}{2}\right]$ da $\cos x \le 0$, $\left[\frac{3\pi}{2},\ 2\pi\right]$da $cosx \ge 0$ ekanligini hisoblaga olib (42.2) formulaga asoslanib topamiz.

$$Q = \int_0^{2\pi} |\cos x|\,dx = \int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \sin x \Big|_{\frac{3\pi}{2}}^{2\pi} =$$

$$= \sin\frac{\pi}{2} \sin 0 - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) + \sin 2\pi - \sin\frac{3\pi}{2} = 1 - 0 - (-1-1) + 0 - (-1) = 4.$$

4-misol. $y = \frac{1}{3}x^2,\ y = 4 - \frac{2}{3}x^2$ parobalalar bilan chegaralangan figuraning yuzi hisoblansin (153-chizma).



153-chizma.

Yechish. Integrallash chegaralari a va b ni $y = \frac{1}{3}x^2$ hamda $y = 4 - \frac{2}{3}x^2$ tenglamalarni birgalikda yechib, ularning kesishish nuqtalari N va M nuqtalarni abssissalarini aniqlash orqali topiladi.

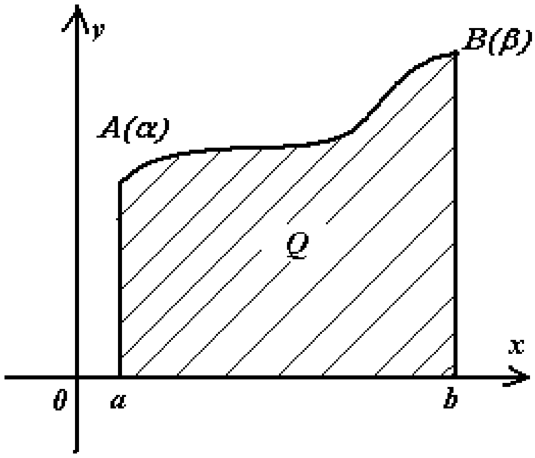
$$\frac{1}{3}x^2 = 4 - \frac{2}{3}x^2, \quad \frac{1}{3}x^2 + \frac{2}{3}x^2 = 4, \qquad x^2 = 4, \qquad x \pm 2.$$

Demak, $a=-2$, $b=2$. (42.3) formulaga binoan topamiz:

$$Q = \int_{-2}^{+2}\left[4 - \frac{2}{3}x^2 - \frac{1}{3}x^2\right]dx = \int_{-2}^{+2}(4 - x^2)dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = 4\cdot 2 - \frac{2^3}{3} - \left(4\cdot(-2) - \frac{(-2)^3}{3}\right) =$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = \frac{32}{3}.$$

Endi egri chiziq parametrik tenglamalari yordamida berilganda egri chiziqli trapetsiyaning yuzini topish formulasini hosil qilamiz.



154-chizma.

Faraz qilaylik egri chiziq $x=\varphi(t)$, $y=\psi(t)$ (42.4) parametrik tenglamalari yordamida berilgan bo'lib bunda $\alpha\le t\le\beta$ va $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo'lsin . Agar (42.2) tenglamalar $[a,b]$ kesmada y=$f(x)$ funksiyani aniqlaydi deb faraz qilsak, u holda egri chiziqli trapetsiyaning yuzi (42.1) formulaga binoan

$$Q=\int_a^b f(x)dx=\int_a^b ydx$$

formula bilan hisoblanadi. Oxirgi integralda o'zgaruvchini almashtiramiz:

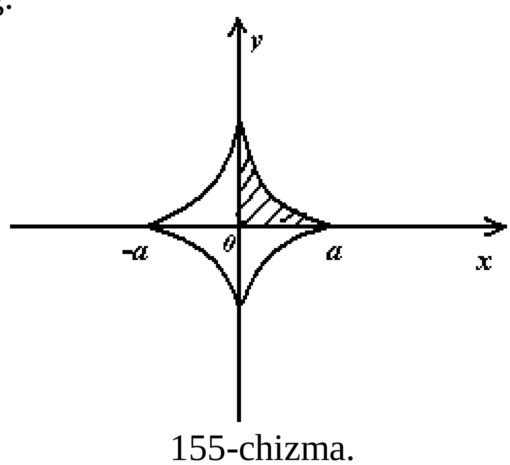
$x=\varphi(t)$, $dx=\varphi'(t)dt$, $\alpha\le t\le\beta$,$y=f(x)=f[\varphi(t)]=\psi(t)$.

Demak,

$$Q=\int_\alpha^\beta \psi(t)\varphi'(t)dt\,. \qquad (42.5)$$

Bu esa parametrik ko'rinishdagi tenglamalari yordamitda berilgan egri chiziqli trapetsiyaning yuzini hisoblash formulasidir.

5-misol. $x=acos^3t$, $y=asin^3t$ astroida (155-chizma) bilan chegaralangan figuraning yuzini hisoblang.



155-chizma.

Yechish. t uchun integrallash chegaralarini $x=acos^3t$ tenglamadan topamiz:

$x=0$ da $acos^3t=0$ $cost=0$, $t=\frac{\pi}{2}$,

$x=a$ da $acos^3t=a$ $cost=0$, $t=1$

Astroidani koordinata o'qlariga nisbatan simmetrikligini hisobga olsak 155-chizmadagi shtrixlangan yuz izlanayotgan yuzning to'rtdan birini tashkil etadi.

Shuning uchun (42.5) formulaga binoan astroida bilan chegaralangan figura yuzining to’rtdan biri uchun quyidagiga ega bo’lamiz.

$$\frac{1}{4}Q=\int_{\frac{\pi}{2}}^{0} a\sin^3 t(a\cos^3 t)'dt=\int_{\frac{\pi}{2}}^{0} a\sin^3 t3a\cos^2 t(\cos t)'dt=-3a^2\int_{\frac{\pi}{2}}^{0}\sin^4 t\cos^2 tdt=3a^2\int_{0}^{\frac{\pi}{2}}(\sin^2 t)^2\cos^2 t$$

$$=3a^2\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2t}{2}\right)^2\cdot\frac{1+\cos 2t}{2}dt=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2t)(1+\cos 2t)(1-\cos 2t)dt=$$

$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos^2 2t)(1-\cos 2t)dt=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2t(1-\cos 2t)dt=$$

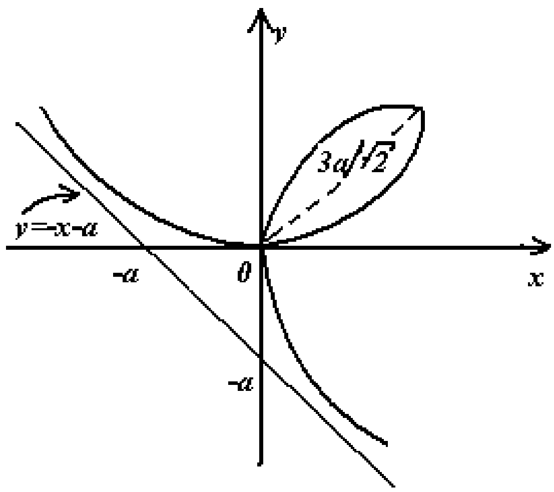
$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2tdt-\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2t\cos 2tdt=\frac{3a^2}{8}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 4t)dt-$$

$$-\frac{3a^2}{8}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2td(\sin 2t)=\frac{3a^2}{16}\left(t-\frac{1}{4}\sin 4t\right)\Big|_0^{\frac{\pi}{2}}-\frac{3a^2}{16}\cdot\frac{\sin^3 2t}{3}\Big|_0^{\frac{\pi}{2}}=$$

$$=\frac{3a^2}{16}\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi\right)-\frac{3a^2}{48}(\sin^3\pi-\sin^3 0)=\frac{3a^2}{16}\left(\frac{\pi}{2}-0\right)-\frac{3a^2}{48}(0-0)=\frac{3a^2\pi}{32}.$$

Bundan $\qquad Q=4\cdot\frac{3a^2\pi}{32}=\frac{3a^2\pi}{8}.$

6-misol. $x=\frac{3at}{1+t^3},\ y=\frac{3at^2}{1+t^3}$ Dekart yaprog’i sirtmogining yuzini hisoblang (156-chizma)



156-chizma.

Yechish.

$$y\cdot dx=\psi(t)\varphi'(t)dt=\frac{3at^2}{1+t^3}\cdot\left(\frac{3at}{1+t^3}\right)'dt=\frac{9a^2t^2}{1+t^3}\cdot\frac{1+t^3-3t^3}{(1+t^3)^2}dt=\frac{9a^2t^2(1-2t^3)}{(1+t^3)^3}dt.$$

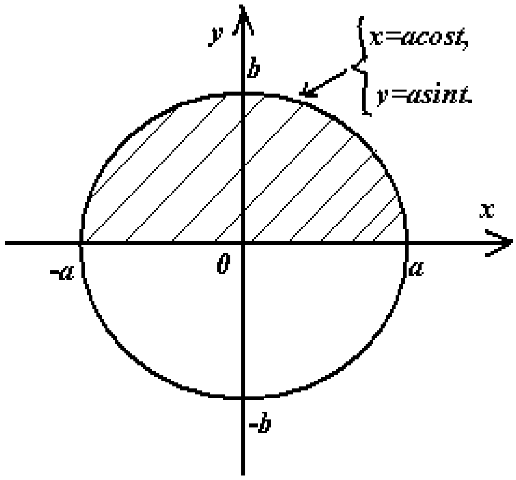
Koordinatalar boshida egri chiziq o’zini o’zi kesadi, ya‘ni koordinatalar boshi egri chiziqning maxsus (qaytish)nuqtasidir. Egri chiziq bu nuqtadan *t=0* da va *t=∞* bo’lganda o’tadi, ya‘ni integrallash chegaralari *0* va ∞ ga teng. Shuning uchun:

$$Q = 9a^2\int_{-\infty}^{0}\frac{t^2(1-2t^3)dt}{(1+t^3)^3}\left|\begin{array}{l} 1+t^3 = z,\ 3t^2dt = dz,\ t^2dt = \frac{1}{3}dz \\ t=0,\ da,\quad z=1, t=\infty,\quad da\quad z=\infty. \end{array}\right| =$$

$$= 9a^2\int_{-\infty}^{1}\frac{(1-2(z-1))dz}{3z^2} = 3a^2\int_{-\infty}^{1}\frac{3-2z}{z^3}dz = 3a^2\int_{-\infty}^{1}(3z^{-3}-\frac{2}{z^2})dz =$$

$$= 3a^2\left(-\frac{3}{2}z^{-2}+\frac{2}{z}\right)\Big|_{-\infty}^{1} = 3a^2\left(-\frac{3}{2}+2\right)-0 = \frac{3a^2}{2}.$$

7-misol. *x=acost, y=bsint* elips bilan chegaralangan figuraning yuzini hisoblang (157-chizma).



157-chizma.

Yechish. Ellips koordinata o’qlariga nisbatan simmetrikligini hisobga olsak 157-chizmadagi shtrixlangan yuz izlanayotgan yuzning yarmini tashkil etadi. Shuning uchun uni hisoblab ikkilantirsak ellips bilan chegaralangan figuraning yuzi hosil bo’ladi. Bu yerda *x* ning qiymati *–a* dan *a* gacha o’zgaradi. U holda *t* ning qiymatini *x=acost* dan aniqlasak u *π* dan *0* gacha o’zgaradi. (42.5) formulaga asosan ellips bilan chegaralangan figura yuzining yarmi uchun quyidagiga ega bo’lamiz:

$$\frac{Q}{2} = \int_{+\pi}^{0} b\sin t(a\cos t)'dt = -ab\int_{\pi}^{0}\sin^2 tdt = -ab\int_{\pi}^{0}\frac{1-\cos 2t}{2} = -\frac{ab}{2}\int_{\pi}^{0}(1-\cos 2t)dt =$$

$$= -\frac{ab}{2}\left(t-\frac{\sin 2t}{2}\right)\Big|_{\pi}^{0} = \frac{ab}{2}\cdot 0+\frac{ab}{2}\left(\pi-\frac{\sin 2\pi}{2}\right) = \frac{ab\pi}{2}.$$

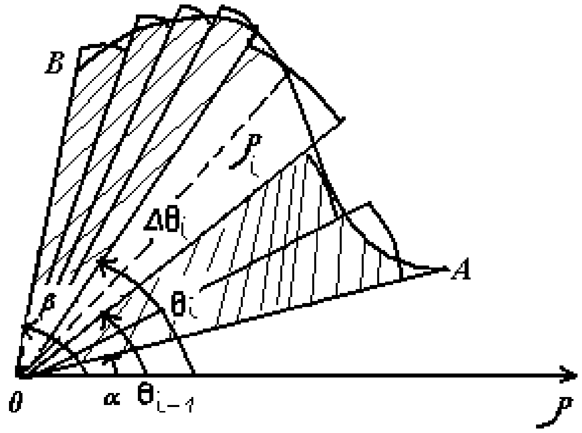
Bundan $Q = ab\pi$ ga ega bo’lamiz.

Xususiy holda *a=b=R* bo’lganda oxirgi tenglikdan doiraning yuzini topish formulasi $Q=\pi R^2$ ni hosil qilamiz.

42.2. Qutb koordinatalar sistemasida egri chiziqli sektorning yuzi

Qutb koordinatalar sistemasida egiri chiziq *ρ=f(θ)* tenglama bilan berilgan bo’lsin, bu yerda *f(θ)*funksiya *[α,β]* kesmada uzluksiz.

$\rho=f(\theta)$egri chiziq, $\theta=\alpha$, $\theta=\beta$ radius-vektorlar bilan chegaralangan OAB sektorning yuzini topamiz (158-chizma).



158-chizma.

Berilgan sektorni $\alpha=\theta_0$, $\theta=\theta_1$, $\theta=\theta_2,\dots$ $\theta=\theta_{n-1}$, $\theta_n=\beta$ radius-vektorlar bilan n ta bo'lakka ajratamiz. O'tkazilgan radius-vektorlar orasidagi burchaklarni $\Delta\theta_1$, $\Delta\theta_2,\dots,\Delta\theta_n$ bilan belgilaymiz. θ_{i-1} bilan θ_i orasidagi biror $\overline{\theta_i}$ burchakka mos radius-vektorning uzunligini $\overline{\rho_i}$ orqali belgilaymiz. Radiusi $\overline{\rho_i}$ va markaziy burchagi $\Delta\theta_i$ bo'lgan doiraviy sektorni qaraymiz. Uni yuzi $\Delta Q_i=\frac{1}{2}\overline{\rho_i}^2\Delta\theta_i$ kabi topilishi ma'lum.

Ushbu yig'indi

$$Q_n=\frac{1}{2}\sum_{i=i}^{n}\overline{\rho}^2\Delta\theta_i=\frac{1}{2}\sum_{i=i}^{n}[f(\overline{\theta_i})]^2\Delta\theta_i.$$

«zinapoyasimon»sektorning yuzini, ya'ni OAB sektorning yuzining taqribiy qiymatini ifodalaydi.

Bu yig'indi $[\alpha,\beta]$ kesmada $\rho^2=[f(\theta)]^2$ uzluksiz funksiyaning integral yig'indisi bo'lganligi sababli u $\lambda=max\Delta\theta_i\to 0$ da

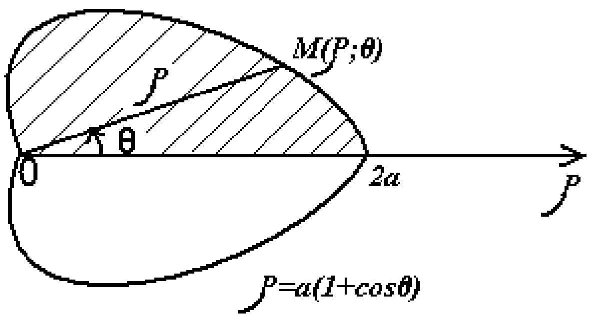
$$\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\theta$$

limitga ega. Shunday qilib OAB sektorning yuzi

$$Q=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\theta=\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta$$

formula yordamida hisoblanar ekan.

8-misol. $\rho=a(1+cos\theta)$ kardioida bilan chegaralangan figuraning yuzini hisoblang (159-chizma).



159-chizma.

Yechish. Figura qutb o’qiga nisbatan simmetrik. Shuning uchun figuraning qutb o’qi ρ dan yuqorida joylashgan qismining yuzini topsak izlanayotgan yuzning yarmi topiladi. θ o’zgaruvchi 0 dan π gacha o’zgarganda ixtiyoriy $M(\rho;\ \theta)$ nuqta kardioidaning ρ o’qdan yuqorida joylashgan qismini chizadi. (42.6) formulaga binoan quyidagiga ega bo’lamiz:

$$\frac{Q}{2}=\frac{1}{2}\int_0^{\pi}[a(1+\cos\theta)]^2d\theta=\frac{a^2}{2}\int_0^{\pi}[a(1+2\cos\theta+\cos^2\theta)d\theta=$$

$$=\frac{a^2}{2}(\theta+2\sin\theta)\Big|_0^{\pi}+\frac{a^2}{2}\int_0^{\pi}\frac{1+\cos 2\theta}{2}d\theta=\frac{a^2}{2}(\pi+2\sin\pi)+\frac{a^2}{4}(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{\pi}=$$

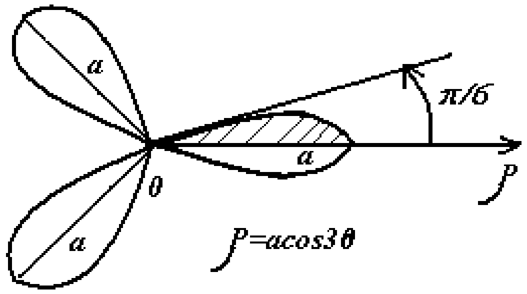
$$=\frac{a^2\pi}{2}+\frac{a^2}{4}(\pi+\frac{1}{2}\sin 2\pi)=\frac{3a^2\pi}{4}.$$

Bundan

$$Q=\frac{3a^2\pi}{2}$$

ga ega bo’lamiz.

9-misol. $\rho=a\ \cos 3\theta$ egri chiziq bilan chegaralangan figuraning yuzini hisoblang (160-chizma).



160-chizma.

Yechish. θ o’zgaruvchi $0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$ qiymatlarni qabul qilganda ρ o’zining eng katta a qiymatiga erishadi. θ o’zgaruvchi $\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6}$ qiymatlarni qabul qilganda egri chiziq qutbdan o’tadi $(\rho=0)$.

160-chizmadagi shrixlangan yuz uch yaproqli gul bilan chegaralangan yuzning oltidan bir qismini tashkil etadi. Shuning uchun (42.6)ga binoan quyidagiga ega bo’lamiz.

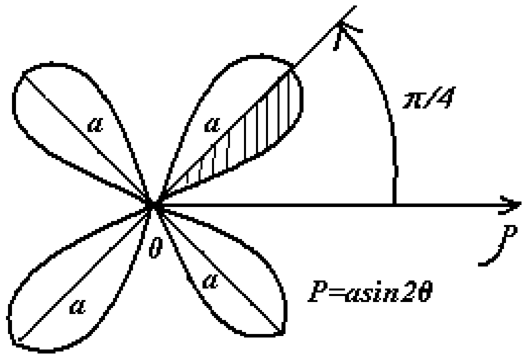
$$\frac{Q}{6}=\frac{1}{2}\int_0^{\frac{\pi}{6}}(a\cos^2 3\theta)^2 d\theta=\frac{a^2}{2}\int_0^{\frac{\pi}{6}}\cos^2 3\theta d\theta=$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{6}}\frac{1+\cos 6\theta}{2}d\theta=\frac{a^2}{4}(\theta+\frac{1}{6}\sin 6\theta)\Big|_0^{\frac{\pi}{6}}=\frac{a^2}{4}\left(\frac{\pi}{6}+\frac{1}{6}\sin\pi\right)=\frac{a^2\pi}{24}.$$

Bundan $Q=\frac{a^2\pi}{4}$

ga ega bo’lamiz.

10-misol. *ρ=asin2θ* egri chiziq bilan chegaralangan figuraning yuzini hisoblang (161-chizma).



161-chizma.

Yechish. *θ* o’zgaruvchi $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ va $\frac{7\pi}{4}$ qiymatlarni qabul qilganda *ρ* o’zining eng katta *a* qiymatiga erishadi. *θ* o’zgaruvchi *0,* $\frac{\pi}{2}$, *π,* $\frac{3\pi}{2}$, qiymatlarni qabul qilganda *(ρ=0)* egri chiziq qutbdan o’tadi.

161-chizmadagi shtrixlangan yuz to’rt yaproqli gul bilan chegaralangan yuzning sakkizdan bir qismini tashkil etadi. Shuning uchun (42.6) formulaga binoan quyidagiga ega bo’lamiz:

$$\frac{Q}{8}=\frac{1}{2}\int_0^{\frac{\pi}{4}}(a\sin 2\theta)^2 d\theta=\frac{a^2}{2}\int_0^{\frac{\pi}{4}}\sin^2 2\theta d\theta=$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{4}}\frac{1-\cos 4\theta}{2}d\theta=\frac{a^2}{4}(\theta-\frac{1}{2}\sin 4\theta)\Big|_0^{\frac{\pi}{4}}=\frac{a^2}{4}\left(\frac{\pi}{4}+\frac{1}{2}\sin\pi\right)=\frac{a^2}{4}\cdot\frac{\pi}{4}=\frac{\pi a^2}{16}.$$

Bundan $Q=8\cdot\frac{\pi a^2}{16}=\frac{\pi a^2}{2}.$

O’z-o’zini tekshirish uchun savollar

1. *y=f(x)≥0, y=0, x=a, x=b(a<b)* chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.
2. *y=f(x), y=0, x=a, x=b* chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.
3. $y=f_1(x)$, $y=f_2(x)$, $(f_1(x)\le f_2(x))$, *x=a,* *x=b* chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.

4. Egri chiziq parametrik tenglamalar bilan berilganda figuraning yuzini topish formulasini yozing.

5. Qutb koordinatalar sistemasida berilgan egri chiziq bilan chegaralangan egri chiziqli sektorning yuzini hisoblash formulasini yozing.

6. Astroida bilan chegaralangan figuraning yuzini topish formulasini yozing.

7. Dekart yaprog'i sirtmogining yuzini topish formulasini yozing.

8. Ellips bilan chegaralangan figura yuzini topish formulasini yozing.

9. Kardioida bilan chegaralangan figura yuzini topish formulasini yozing.

10. $\rho=a\ \cos 3\theta$ egri chiziq bilan chegaralangan figuraning yuzini topish formulasini keltirib chiqaring.

11. $\rho=\alpha\ \sin 2\theta$ egri chiziq bilan chegaralangan figuraning yuzini topish formulasini keltirib chiqaring.

Mustaqil yechish uchun mashqlar

Quyidagi chiziqlar bilan chegaralangan figuraning yuzlari topilsin.

1. $y=\ln x,\ \ x=e,\ \ y=0.$ Javob: $1.$
2. $y=e^x,\ \ y=e^{-x},\ \ x=2.$ Javob: $(e-e^{-1})^2$.
3. $y=x(x-1)(x-2),\ y=0.$ Javob: $\frac{1}{2}.$
4. $y=x^2+4x,\ \ y=x+4.$ Javob: $20\frac{5}{6}.$
5. $y=\dfrac{a^3}{x^2+a^2},\ \ y=0.$ Javob: πa^2.
6. $x=a(t-\sin t),\ \ y=a(1-\cos t)$ sikloidaning bir arkasi va $0x$ o'q bilan chegaralangan figuraning yuzini hisoblang. Javob: $3\pi a^2$.

Quyidagi chiziqlar bilan chegaralangan figura yuzini hisoblang.

7. $\rho=a\sin 2\theta.$ Javob: $Q=\dfrac{\pi a^2}{4}.$
8. $\rho=a\cos 2\theta.$ Javob: $Q=\dfrac{\pi a^2}{2}.$

Egri chiziqning eng katta va eng kichik qo'shni radius-vektorlari orasidagi figuraning yuzini toping.

9. $\rho=3-\cos 2\theta$. Javob: $Q=\dfrac{19\pi}{8}.$
10. $\rho=2+\sin 3\theta.$ Javob: $Q=\dfrac{3\pi}{4}.$

43-mavzu. Mavzu: Egri chiziq yoyining uzunligi

Reja:

1. Dekart koordinatalar sistemasida egri chiziq yoyining uzunligi.
2. Qutb koordinatalar sistemasida egri chiziq yoyining uzunligi.

Adabiyotlar: 1,4,7,9,10,12

Tayanch iboralar: siniq chiziq, zveno, yoy uzunligi, integral yig'indi, kubik parabola, sikloida, parametrik, qutb tenglama.

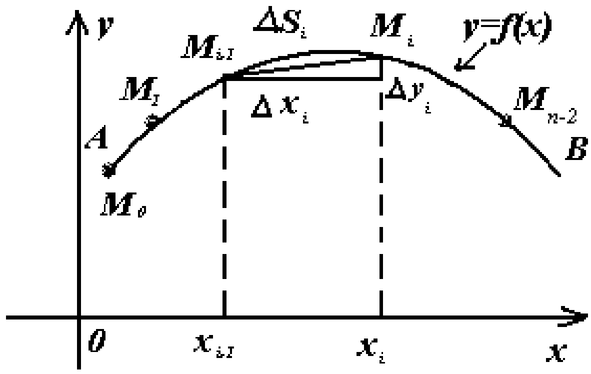
43.1. Dekart koordinatalar sistemasida egri chiziq yoyining uzunligi

Egri chiziq *y=f(x)* tenglama bilan berilganda uning *x=a* va *x=b (a<b)* vertikal to'g'ri chiziqlar orasidagi *AB* yoyining uzunligini topamiz, bunda *y=f(x)* funksiya *[a,b]* kesmada uzluksiz deb faraz qilamiz. *AB* yoyda abssissalari $a=x_0$, $x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_n=b$ bo'lgan $A, M_1, M_2, \ldots, M_{i-1}, M_i, \ldots, B$ nuqtalarni olamiz va AM_1, $M_1M_2, \ldots$, $M_{i-1}M_i, \ldots$, $M_{n-1}B$ vatarlarni o'tkazamiz hamda ularning uzunliklarini mos ravishda ΔS_1, $\Delta S_2, \ldots$, ΔS_n bilan belgilaymiz.

Bu holda *AB* yoyga ichki chizilgan $AM_1M_2, \ldots, M_{n-1}B$ siniq chiziq hosil bo'ladi. Siniq chiziqning uzunligi

$$S_n = \sum_{i=1}^{n} \Delta S_i$$

ga teng bo'ladi (162-chizma).



162-chizma.

AB yoyning uzunligi deb unga ichki chizilgan siniq chiziqning eng katta zvenosi uzunligi nolga intilgandagi

$$S = \lim_{\max \Delta S_i \to 0} \sum_{i=1}^{n} \Delta S_i \quad (43.1)$$

limitga aytiladi.

Endi *f(x)* funksiya *[a,b]* kesmada uzluksiz $f'(x)$ hosilaga ega bo'lganda yoy uzunligini hisoblash formulasini chiqaramiz.

$\Delta y_i = f(x_i) - f(x_{i-1})$, $\Delta x_i = x_i - x_{i-1}$ deb belgilasak.

$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i$$

bo'ladi. Lagranj teorimasiga asosan $\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(z_i)$, bunda $x_{i-1} < z_i < x_i$. Demak $\Delta S_i = \sqrt{1 + [f'(z_i)]^2}\ \Delta x_i$ va $S_n = \sum_{i=1}^{n} \sqrt{1 + [f'(z_i)]^2}\ \Delta x_i$ bo'ladi.

S_n *[a,b]* kesmada uzluksiz $\sqrt{1 + [f'(x)]^2}$ funksiyaning integral yig'indisi bo'lganligi sababli $\lambda = \max \Delta x_i \to 0$ da uning limiti mavjud va quyidagi aniq integralga teng.

$$S = \lim_{\max \Delta S_i \to 0} \sum_{i=1}^{n} \Delta S_i = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1+[f'(z_i)]^2}\,\Delta x_i = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

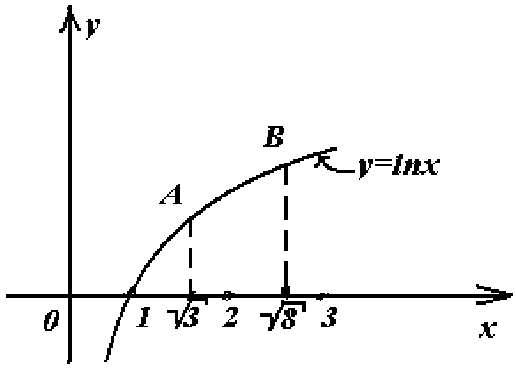
Shunday qilib AB yoy uzunligini hisoblash uchun

$$S = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+y'^2}\,dx \qquad (43.2)$$

formulani hosil qildik.

1-misol. $y = \ell nx$ egri chiziq yoyining $x=\sqrt{3}$ dan $x=\sqrt{8}$ gacha bo'lgan qismi uzunligini hisoblang.

Yechish. $1+y'^2=1+(\ell nx)'^2=1+\frac{1}{x^2}=\frac{1+x^2}{x^2}.$ (43.2) formulaga binoan quyidagiga
ega bo'lamiz:



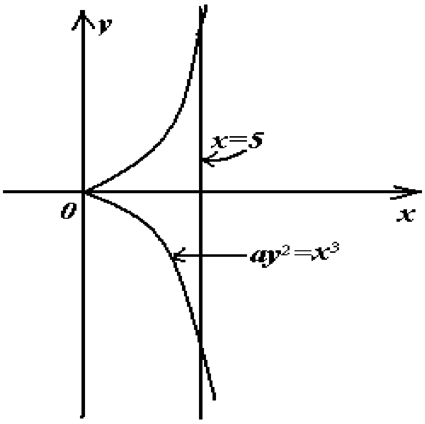
163-chizma.

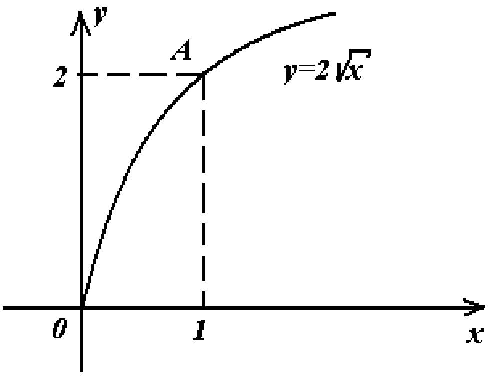
$$S = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}dx \left| \begin{matrix} 1+x^2 = t^2, x=\sqrt{t^2-1}, dx=\dfrac{tdt}{\sqrt{t^2-1}} \\ x=\sqrt{3}, da\; t=2; x=\sqrt{8} da, t=3 \end{matrix} \right| = \int_2^3 \frac{t^2}{t^2-1}dt =$$

$$= \int_2^3 \frac{t^2-1+1}{t^2-1}dt = \int_2^3 \left(1+\frac{1}{t^2-1}\right)dt = \left(t+\frac{1}{2}\ell n\left|\frac{t-1}{t+1}\right|\right)\Big|_2^3 = 3+\frac{1}{2}\ell n\frac{2}{4}-2-\frac{1}{2}\ell n\frac{1}{3} =$$

$$=1+\frac{1}{2}\left(\ell n\frac{1}{2}-\ell n\frac{1}{3}\right)=1+\frac{1}{2}\ell n\frac{1}{2}\cdot\frac{3}{1}=1+\frac{1}{2}\ell n\frac{3}{2}.$$

2-misol. $ay^2=x^3$ $(a>o)$ yarim kubik parabola yoyining koordinatalar boshidan abssissasi $x=5a$ nuqtagacha bo'lgan qismining uzunligini hisoblang (164-chizma).





164-chizma. 165-chizma.

Yechish. $(ay^2)'=(x^2)'$, $2ay\cdot y'=3x^2$. $y'=\dfrac{3x^2}{2ay}$; $y'^2=\dfrac{9x^4}{4a^2y^2}=\dfrac{9x^4}{4a^2\cdot\dfrac{x^3}{a}}=\dfrac{9x}{4a}$.

(43.2) formulaga binoan quyidagiga ega bo'lamiz:

$$S=\int_0^{5a}\sqrt{1+y'^2}\,dx=\int_0^{5a}\sqrt{1+\frac{9}{4a}x}\,dx=\int_0^{5a}\left(1+\frac{9}{4a}x\right)^{\frac{1}{2}}dx=\frac{4a}{9}\cdot\frac{(1+\frac{9}{4a}x)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{5a}=$$

$$=\frac{8a}{27}\left[\left(1+\frac{9}{4a}\cdot 5a\right)^{\frac{3}{2}}-1\right]=\frac{8a}{27}\left[\left(\frac{49}{4}\right)^{\frac{3}{2}}-1\right]=\frac{8a}{27}\left[\left(\left(\frac{7}{2}\right)^2\right)^{\frac{3}{2}}-1\right]=\frac{8a}{27}\left(\frac{7^3}{2^3}-1\right)=\frac{8a}{27}\cdot\frac{7^3-2^3}{8}=\frac{335}{27}a.$$

3-misol. $y=2\sqrt{x}$ parabola yoyining $x=0$ dan $x=1$ gacha bo'lgan qismi OA yoyining uzunligini hisoblang (165-chizma).

Yechish. $y'=2\cdot\dfrac{1}{2\sqrt{x}}=\dfrac{1}{\sqrt{x}}, 1+y'^2=1+\dfrac{1}{x}=\dfrac{x+1}{x}=\dfrac{(x+1)^2}{x(x+1)}=\dfrac{(x+1)^2}{x^2+x}$.

formulaga binoan

$$S=\int_0^1\sqrt{\frac{(x+1)^2}{x^2+x}}dx=\int_0^1\frac{x+1}{\sqrt{x^2+x}}dx=\frac{1}{2}\int_0^1\frac{2x+1+1}{\sqrt{x^2+x}}dx=\frac{1}{2}\int_0^1\frac{2x+1}{\sqrt{x^2+x}}dx+\frac{1}{2}\int_0^1\frac{dx}{\sqrt{x^2+x}}=$$

$$=\frac{1}{2}\int_0^1\frac{(x^2+x)'dx}{\sqrt{x^2+x}}+\frac{1}{2}\int_0^1\frac{dx}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}=\int_0^1\frac{d(x^2+x)}{2\sqrt{x^2+x}}+\frac{1}{2}\int_0^1\frac{dx}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}}=$$

$$=\sqrt{x^2+x}\,\Big|_0^1+\frac{1}{2}\cdot\ell n\left|x+\frac{1}{2}+\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\right|\Big|_0^1=\sqrt{2}+\frac{1}{2}\ln\left(\frac{3}{2}+\sqrt{2}\right)-\frac{1}{2}\ln\frac{1}{2}=\sqrt{2}+\frac{1}{2}\cdot\ell n(\frac{3}{2}+\sqrt{2})\cdot 2=$$

$$=\sqrt{2}+\frac{1}{2}\ell n(3+2\sqrt{2}).$$

Endi egri chiziq tenglamasi *x=φ(t), x=Ψ(t), α≤t≤β* (43.3) parametrik ko'rinishda berilgan bo'lsa shu egri chiziq yoyining uzunligini topamiz. *φ(x), Ψ(x)* funksiyalar *[α, β]* kesmada uzluksiz va uzluksiz hosilalarga ega deb faraz qilamiz. *φ(α)=a, φ(β)=b* bo'lsin. (43.3) tenglamalar *[a,b]* kesmada biror *y=f(x)* funksiyani aniqlaydi deb faraz qilsak izlanayotgan yoy uzunligi

$$S=\int_a^b\sqrt{1+y'^2}dx$$

formula yordamida topiladi (43.2formula). Bu integralda *x=φ(t), dx=φ′(t)dt* almashtirishni bajarsak $y'=\frac{\psi'(t)}{\varphi'(t)}$ ekanini hisobga olib quyidagiga ega bo'lamiz:

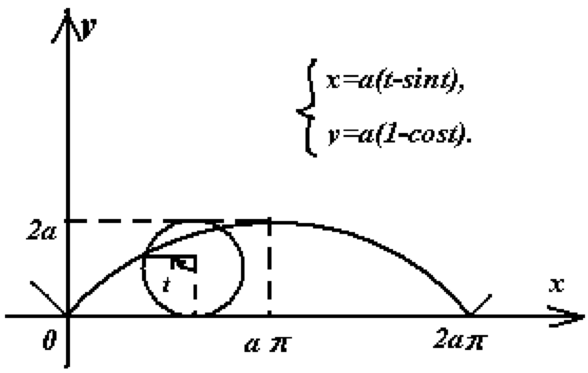
$$S=\int_\alpha^\beta\sqrt{1+\left[\frac{\psi'(t)}{\varphi'(t)}\right]^2}\varphi'(t)dt \quad \text{yoki} \quad S=\int_\alpha^\beta\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt.$$

Shunday qilib (43.3) parametrik tenglamalari yordamida berilgan egri chiziq yoyining uzunligi $S=\int_\alpha^\beta\sqrt{x'^2+y'^2}dt=\int_\alpha^\beta\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt$ (43.4)

formula yordamida topilar ekan.

4-misol.*x=a(t-sint), y=a(1-cost)* sikloida bitta arkasining uzunligi topilsin (166-chizma).

Yechish. To'g'ri chiziq bo'ylab harakatlanayotgan g'ildirakning (aylananing) berilgan aniq nuqtasini traektoriyasi sikloida deb ataluvchi egri chiziqni chizadi. G'ildirak to'liq bir marta aylanganda uning aniq nuqtasini chizgan chizigi sikloidaning bitta arkasini tashkil etadi. G'ildirakni radiusi *a* hamda harakat boshlanganga qadar g'ildirakning aniq nuqtasi koordinatalar boshida bo'lgan deb faraz qilinsa sikloidani ifodalovchi egri chiziq 166-chizmada tasvirlangan.



166-chizma.

$x'=a(1-cost),\ y'=asint,\ x'^2+y'^2=a^2(1-cost)^2+a^2sin^2t=a^2(1-2cost+cos^2t+sin^2t)==a^2(2-2cost)=2a^2(1-cost)=4a^2\sin^2\frac{t}{2}=(2a\sin\frac{t}{2})^2.$

$x=0$ da $t=0$ va $x=2a\pi$ da $t=2\pi$.
(43.4) formulaga binoan:

$$S=\int_0^{2\pi}\sqrt{x'^2+y'^2}\,dt=\int_0^{2\pi}\sqrt{(2a\sin\frac{t}{2})^2}\,dt=2a\int_0^{2\pi}\sin\frac{t}{2}\,dt=2a\cdot 2(-\cos\frac{t}{2})\Big|_0^{2\pi}=$$

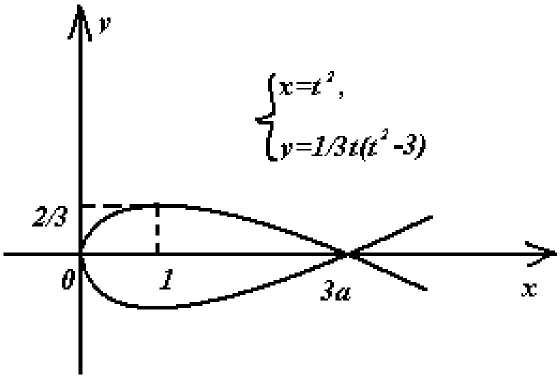
$$=4a(-\cos\pi+\cos o)=8a.$$

5-misol. $x=t^2,\ y=\frac{1}{3}t(t^2-3)$ egri chiziqning $0x$ o'q bilan kesishish nuqtalari orasidagi yoyi uzunligini hisoblang (167-chizma).

Yechish. t ning qanday qiymatlarida egri chiziq Ox o'qni kesib o'tishini aniqlaymiz:

$y=\frac{1}{3}t(t^2-3)=0$ tenglamadan

$t_1=0$, $t_2=\pm\sqrt{3}$ kelib chiqadi. Bularni $x=t^2$ ga qo'ysak $x_1=0$ va $x_2=3$ egri chiziq bilan Ox o'qni kesishish nuqtalarining abssissalari hosil bo'ladi. Yoyning yarmi uchun integrallash chegaralari $\alpha=0$ va $\beta=\sqrt{3}$ bo'ladi.



167-chizma.

$$x'=(t^2)'=2t,\qquad y'=(\frac{1}{3}t(t^2-3))'=\frac{1}{3}(t^3-3t)'=\frac{1}{3}(3t^2-3)=t^2-1.$$

$$\sqrt{x'^2+y'^2}=\sqrt{(2t)^2+(t^2-1)^2}=\sqrt{4t^2+t^4-2t^2+1}=\sqrt{(t^2+1)^2}=t^2+1$$

Demak, (43.4) formulaga binoan

$$\frac{S}{2}=\int_0^{\sqrt{3}}(t^2+1)dt=\left(\frac{t^3}{3}+t\right)\Big|_0^{\sqrt{3}}=\frac{(\sqrt{3})^3}{3}+\sqrt{3}=2\sqrt{3},$$

bundan $s=4\sqrt{3}$ hosil bo'ladi.

6-misol. $\begin{cases}x=e^t\cos t,\\ y=e^t\sin t\end{cases}$ egri chiziq yoyining $t=0$ dan $t=\ell n\,\pi$ gacha qismining uzunligini hisoblang.

Yechish. $x'=e^t\cos t-e^t\sin t$, $y'=e^t\sin t+e^t\cos t$,

$x'^2+y'^2=e^{2t}(\cos t-\sin t)^2+e^{2t}(\sin t+\cos t)^2=e^{2t}(\cos^2 t-2\cos t\sin t+\sin^2 t+$
$+\sin^2 t+2\sin t\cos t+\cos^2 t)=e^{2t}(2\cos^2 t+2\sin^2 t)=2e^{2t}(\cos^2 t+\sin^2 t)=2e^{2t}$.

(43.1) formulaga binoan quyidagiga ega bo'lamiz:

$$S=\int_0^{\ell n\pi}\sqrt{2e^{2t}}\,dt=\sqrt{2}\int_0^{\ell n\pi}e^t\,dt=\sqrt{2}e^t\Big|_0^{\ell n\pi}=\sqrt{2}(e^{\ell n\pi}-e^0)=\sqrt{2}(\pi-1).$$

($a^{\ell og_a b}=b$ ayniyatdan foydalandik).

43.2. Qutb koordinatalar sistemasida egri chiziq yoyining uzunligi

Qutb koordinatalar sistemasida egri chiziq $\rho=f(\theta)$ tenglama bilan berilgan bo'lsin, bu yerdagi $f(\theta)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz hosilaga ega bo'lgan funksiya, ρ-qutb radiusi, θ-qutb burchagi. Shu egri chiziq yoyining uzunligini topish formulasini hosil qilamiz. Qutb koordinatalaridan dekart koordinatalariga o'tish formulasi $x=\rho\cos\theta$, $y=\rho\sin\theta$ ga ρ o'rniga $f(\theta)$ ni qo'ysak $x= f(\theta)\cos\theta$, $y= f(\theta)\sin\theta$ tenglamalar hosil bo'ladi. Bu tenglamalarga egri chiziqning parametrik tenglamalari deb qarab yoy uzunligini hisoblash formulasi (43.4) dan foydalanamiz:

$x'_\theta=(f(\theta)\cos\theta)'= f'(\theta)\cos\theta- f(\theta)\sin\theta$, $y'_\theta = (f(\theta)\sin\theta)'= f'(\theta)\sin\theta+ f(\theta)\cos\theta$,

$x'^2_\theta+ y'^2_\theta=(f'(\theta)\cos\theta- f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+ f(\theta)\cos\theta)^2= f'^2(\theta)\cos^2\theta-$

$-2f'(\theta) f(\theta) \cos\theta\sin\theta+ f^2(\theta)\sin^2\theta+f'^2(\theta)\sin^2\theta+2f'(\theta) f(\theta)\cos\theta\sin\theta+ f^2(\theta)\cos^2\theta= =f'^2(\theta)(\cos^2\theta+ \sin^2\theta)+ f^2(\theta)(\cos^2\theta+ \sin^2\theta)= f'^2(\theta)+ f^2(\theta)= \rho'^2+\rho^2$.

Demak, $$S=\int_a^B\sqrt{\rho'^2+\rho^2}\,d\theta. \qquad (43.5)$$

Bu qutb koordinatalar sistemasida berilgan egri chiziqning yoy uzunligini topish formulasidir.

7-misol. $\rho=a(1+\cos\theta)$ kardioidaning uzunligini hisoblang (159-chizma).

Yechish. Kardioida qutb o'qiga nisbatan simmetrik joylashganligi sababli qutb burchagi θ 0 dan π gacha o'zgarsa (43.5) formula yordamida kardioida uzunligining yarmi topiladi:

$$\frac{S}{2}=\int_0^{\pi}\sqrt{[a(1+\cos\theta)']^2+[a(1+\cos\theta)]^2}\,d\theta=\int_0^{\pi}\sqrt{a^2(\sin^2\theta+1+2\cos\theta+\cos^2\theta)}\,d\theta=$$

$$=a\int_0^{\pi}\sqrt{2(1+\cos\theta)}\,d\theta=a\int_0^{\pi}\sqrt{4\cos^2\frac{\theta}{2}}\,d\theta=a\int_0^{\pi}2\cos\frac{\theta}{2}\,d\theta=2a\cdot 2\sin\frac{\theta}{2}\Big|_0^{\pi}=$$

$$=4a(\sin\frac{\pi}{2}-\sin 0)=4a;\ S=2\cdot 4a=8a.$$

8-misol. $\rho=a\theta$ Arximed spirali birinchi o'rami yoyining uzunligini hisoblang (168-chizma).

168-chizma.

Yechish. $\rho'=a, \sqrt{\rho'^2+\rho^2}=\sqrt{a^2+a^2\theta^2}=a\sqrt{1+\theta^2}$.

(43.2) formulaga binoan:

$$S=a\int_0^{2\pi}\sqrt{1+\theta^2}\,d\theta\left|\begin{matrix}\sqrt{1+\theta^2}=u, d\theta=dv\\ du=\dfrac{\theta d\theta}{\sqrt{1+\theta^2}}, v=\theta\end{matrix}\right|=a\theta\sqrt{1+\theta^2}\Big|_0^{2\pi}-a\int_0^{2\pi}\frac{\theta^2d\theta}{\sqrt{1+\theta^2}}=$$

$$=a2\pi\sqrt{1+4\pi^2}-a\int_0^{2\pi}\frac{1+\theta^2-1}{\sqrt{1+\theta^2}}d\theta=a2\pi\sqrt{1+4\pi^2}-a\int_0^{2\pi}\sqrt{1+\theta^2}\,d\theta+a\int_0^{2\pi}\frac{d\theta}{\sqrt{1+\theta^2}}=$$

$$=a2\pi\sqrt{1+4\pi^2}-S+a\ell n(\theta+\sqrt{1+\theta^2})\Big|_0^{2\pi}=a2\pi\sqrt{1+4\pi^2}-S+a\ell n(2\pi+\sqrt{1+4\pi^2}).$$

Bundan

$$2S=2a\pi\sqrt{1+4\pi^2}+a\ell n(2\pi+\sqrt{1+4\pi^2})$$

yoki

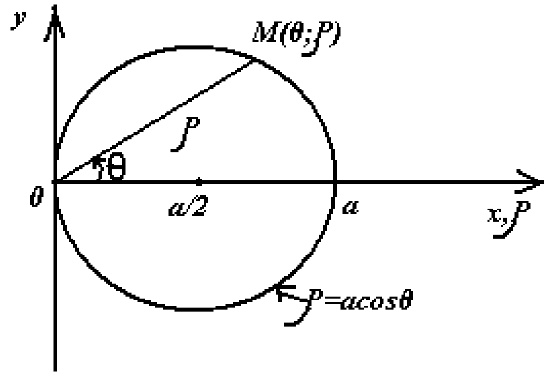
$$S = a\pi\sqrt{1+4\pi^2} + \frac{a}{2}\ell n(2\pi + \sqrt{1+4\pi^2}).$$

9-misol. $\rho = a\cos\theta$ aylana yoyining uzunligini hisoblang.

Yechish. Avvalo berilgan egri chiziqni aylana ekanligiga ishonch hosil qilish uchun tenglamani ρ ga ko'paytiramiz. U holda $\rho^2 = a\,\rho\cos\theta$ tenglik hosil bo'ladi. Qutb va dekart koordinatalari orasidagi bog'lanish

$\rho^2 = x^2 + y^2$, $x = \rho\cos\theta$

ni hisobga olsak $x^2+y^2=ax$, $x^2-ax+y^2=0$ yoki $(x-\frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$ tenglamaga ega bo'lamiz. Bu tenglama markazi $(\frac{a}{2};\ 0)$ nuqtada bo'lib radiusi $\frac{a}{2}$ ga teng aylana tenglamasidir (169-chizma).



169-chizma.

Aylananing qutb o'qidan yo'qorida joylashgan yarmining uzunligini hisoblaymiz:

$$\frac{S}{2} = \int_0^{\frac{\pi}{2}} \sqrt{\rho^2 + \rho'^2}\,d\theta = \int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\,d\theta = a\int_0^{\frac{\pi}{2}} d\theta = a\theta\,\Big|_0^{\frac{\pi}{2}} = a\frac{\pi}{2}.$$

Bundan $S = a\cdot\frac{\pi}{2}\cdot 2 = a\pi$ o'zimizga ma'lum aylanani uzunligini topish formulasiga ega bo'lamiz.

O'z-o'zini tekshirish uchun savollar

1. Yoy uzunligini ta'riflang.
2. $y=f(x)$ tenglama bilan berilgan egri chiziq yoyining uzunligini topish formulasini yozing.
3. Egri chiziq parametrik tenglamalar bilan berilganda uning yoyini uzunligini topish formulasini yozing.
4. Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligini topish formulasini yozing.
5. Sikloida nima va uning uzunligi qanday topiladi?
6. Kardioidaning uzunligini hisoblash formulasini chiqaring.
7. Arximed spirali birinchi o'rami uzunligini hisoblash formulasini chiqaring.
8. Aylana uzunligini hisoblash formulasini chiqaring.
9. Ellips uchun yoy uzunligini topish formulasini chiqarishga harakat qiling.
10. $y= f(x)$ egri chiziq yoyining uzunligini hisoblash uchun chiqarilgan formulalarda $f(x)$ funksiya qanday shartlarni qanoatlantiradi?

Mustaqil yechish uchun mashqlar

1. $y^2=\frac{1}{9}(x+1)^3$ egri chiziq yoyining $0y$ o'q ajratgan qismining uzunligini hisoblang. Javob: $\frac{2}{3}(5\sqrt{5}-8)$.

2. $y=\ell n(1-x^2)$ egri chiziq yoyining $x=0$ dan $x=\frac{1}{3}$ gacha qismining uzunligini hisoblang. Javob: $\ell n\, 2-\frac{1}{3}$.

3. $y^2=4(x-1)$egri chiziq yoyining $x=1$ dan $x=2$ gacha bo'lgan qismining uzunligini hisoblang. Javob: $\sqrt{3}+\frac{1}{2}\ell n\frac{9}{5}$.

4. $\left.\begin{matrix} x=4\sin t+3\cos t, \\ y=4\cos t-3\sin t \end{matrix}\right\}, 0\le t\le \pi$ egri chiziq yoyining uzunligini hisoblang. Javob: 5π.

5. $\begin{cases} x=t^2-1, \\ y=\frac{t^3}{3}-t. \end{cases}$ egri chiziq sirtmog'i uzunligini hisoblang. Javob : $4\sqrt{3}$.

6. $\rho=a\sin\theta$ egri chiziq yoyi uzunligini hisoblang. Javob: πa.

<u>44-ma'ruza.</u>Mavzu: Aniq integral yormida jismning hajmi va sirtini topish

Reja:

1. Aylanish jismining hajmi.
2. Aylanish jismining sirti.

Adabiyotlar: 1,4,7,9,10,12.

Tayanch iboralar: hajm, yon sirt, to'la sirt, qutb o'qi, qutb burchagi, egri chiziqli trapetsiya, egri chiziqli sektor, to'g'ri to'rtburchak.

44.1. Aylanish jismining hajmi

$y=f(x)$ funksiya $[a,b]$ kesmada uzluksiz va nomanfiy funksiya bo'lsin. $y=f(x)$ egri chiziq, $0x$ o'q va $x=a$, $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan $aABb$ egri chiziqli trapetsiyani $0x$ o'q atrofida aylanish natijasida hosil bo'lgan aylanish jismininining hajmini topamiz. $[a,b]$ kesmani

$$a=x_o< x_1< x_2<\ldots <x_{i-1}< x_i<\ldots <x_{n-1}<x_n=b$$

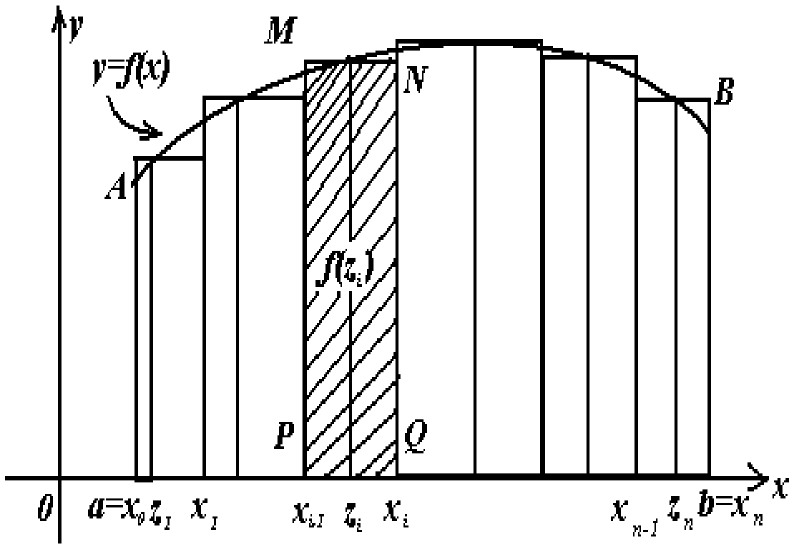
nuqtalar yordamida n ta ixtiyoriy

$$[x_o,x_1],[x_1,x_2],\ldots,[x_{i-1}, x_i],\ldots [x_{n-1},x_n]$$

kesmalarga ajratamiz. Har bir $[x_{i-1}, x_i]$ $(i=\overline{1,n})$ kesmachada ixtiyoriy z_i nuqtani olib funksiyaning bu nuqtadagi qiymati $f(z_i)$ ni hisoblaymiz.

Keyin asosi $\Delta x_i=x_i-x_{i-1}$ bo'lib balandligi $f(z_i)$ bo'lgan $PMNQ$ to'g'ri to'rtburchak yasaymiz.

Bu to'g'ri to'rtburchak $0x$ o'q atrofida aylanganda asosining radiusi $f(z_i)$ bo'lib balandligi Δx_i bo'lgan doiraviy silindr hosil bo'ladi. Bu silindrning hajmi $\vartheta_i=\pi R^2H=\pi f^2(z_i)\ \Delta x_i$ formula yordamida topilishi ravshan. Barcha n ta silindrlar hajmlarining yig'indisi qaralayotgan aylanish jismi hajmi V_x ning taqribiy qiymatini beradi, ya'ni $V_x\approx\sum_{i=i}^{n}\pi f^2(z_i)\Delta x_i$



170-chizma.

Ikkinchi tomondan bu yig'indi $[a,b]$ kesmada uzluksiz $\pi f^2(x)$ funksiya uchun integral yig'indi bo'lganligi uchun $y\ \lambda=\max\Delta x_i\to 0$ da $\pi\int_a^b f^2(x)dx$ aniq integralga teng limitga ega.

Shunday qilib aylanish jismining hajmi

$$V_x=\pi\int_a^b f^2(x)dx \quad (44.1)$$

formula yordamida topilar ekan.

Endi $aABb$ egri chiziqli trapetsiyani $0y$ o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmini topamiz. $PMNQ$ to'g'ri to'rtburchak $0y$ o'q atrofida aylanish natijasida hosil bo'lgan silindrning hajmi asosida radiusi x_i balandligi $f(z_i)$ bo'lgan silindrning hajmidan asosining radiusi x_{i-1} balandligi $f(z_i)$ bo'lgan silindr hajmining ayrilganiga teng, ya'ni

$$\overline{V_i}=\pi x_i^2 f(z_i)-\pi x_{i-1}^2 f(z_i)=\pi f(z_i)(x_i^2-x_{i-1}^2)=\pi f(z_i)(x_i+x_{i-1})(x_i-x_{i-1})=$$
$$=\pi f(z_i)(x_i+x_{i-1})\Delta x_i\,.$$

$\sum_{i=i}^{n} f(z_i)(x_i+x_{i-1})\Delta x_i\,.$ yig'indi $aABb$ egri chiziqli trapetsiya $0y$ o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmi V_y ni taqribiy qiymatini beradi.

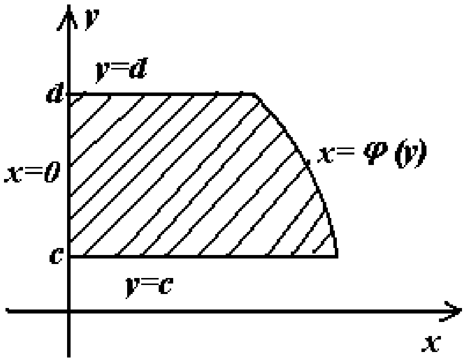
Demak $\quad V_y\approx\sum_{i=i}^{n} f(z_i)(x_i+x_{i-1})\Delta x_i\,.$

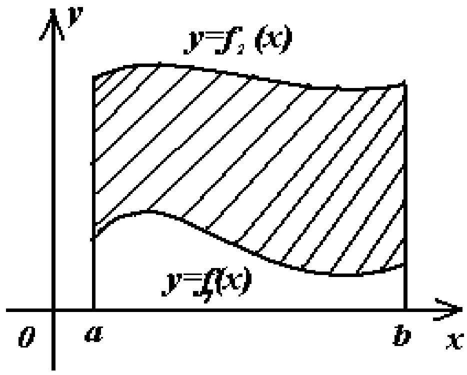
Bu yig'indi $[a,b]$ kesmada uzluksiz $2\pi xf(x)$ funksiya uchun integral yig'indi bo'lganligi uchun ta $x\Delta x_i \to 0$ aniq limitga ega. $x_{i-1}<z_i<x_i$ bo'lib $\Delta x_i = x_{i-1}-x_i$ 0 ga intilganda $x_{i-1}+x_i$ yig'indi $2z_i$ ga intiladi.

Shunday qilib

$$V_y = \lim_{\max \Delta x_i \to 0} \sum_{i=i}^{n} \pi f(z_i)(x_i + x_{i-1})\Delta x_i = \lim_{\max \Delta x_i \to 0} \sum_{i=i}^{n} 2\pi f(z_i) z_i \Delta x_i = 2\pi \int_a^b f(x)xdx. \quad (44.2)$$

formulaga ega bo'lamiz.





171-chizma. 172-chizma.

Agar egri chiziqli trapetsitsiya $y=\varphi(y)$ egri chiziq, $0y$ o'qi $y=c, y=d$ $(c<d)$ to'g'ri chiziqlar bilan chegaralangan bo'lsa (171-chizma), u holda bu figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V_y = \pi \int_c^d x^2 dy. \quad (44.3)$$

formula yordamida hisoblanadi.

Agar $y=f_1(x)$, $y=f_2(x)$, $(f_1(x)\leq f_2(x))$ egri chiziqlar va $x=a$ hamda $x=b$ to'g'ri chiziqlar bilan chegaralangan figura $0x$ o'q atrofida aylanayotgan bo'lsa (172-chizma), u holda hosil bo'lgan aylanish jismining hajmi

$$V_x = \pi \int_a^b [f_2^2(x) - f_1^2(x)]dx \quad (44.4)$$

formula yordamida hisoblanadi. Agar bu figura $0y$ o'q atrofida aylanayotgan bo'lsa, u holda aylanish jismining hajmi

$$V_y = \pi \int_a^b [f_2(x) - f_1(x)]xdx. \quad (44.5)$$

formula bo'yicha hisoblanadi.

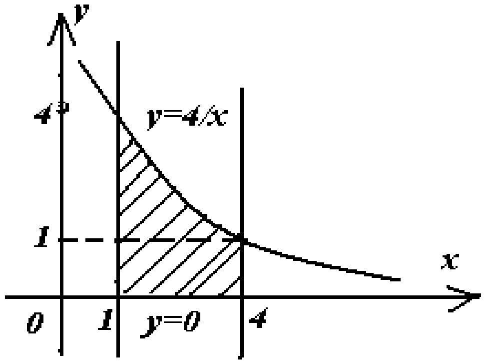
Agar $y=f(x)$ egri chiziq parametrik yoki qutb koordinatalar sistemasida berilgan bo'lsa, u holda yuqorida keltirilgan barcha formulalaroda integrallash o'zgaruvchisini tegishlicha almashtirish kerak.

Qutb koordinatalar sistemasida $\rho=f(\theta)$ egri chiziq hamda qutb o'qi bilan α va $\beta(\alpha<\beta)$ burchak tashkil etuvchi $\theta=\alpha$, $\theta=\beta$ nurlar bilan chegaralangan egri chiziqli sektorni qutb o'qi atrofida aylanishi natijasida hosil bo'lgan jismning hajmi

$$V_\rho = \frac{2}{3}\pi \int_\alpha^\beta \rho^3 \sin\theta d\theta \quad (44.6)$$

formula yordamida topilishni ta'kidlab o'tamiz.

1-misol. *xy=4, x=1, x=4, y=0* chiziqlar bilan chegaralangan *0x* va *0y* o'qlari atrofida aylanishidan hosil bo'lgan jismning hajmini toping (173-chizma).



173-chizma.

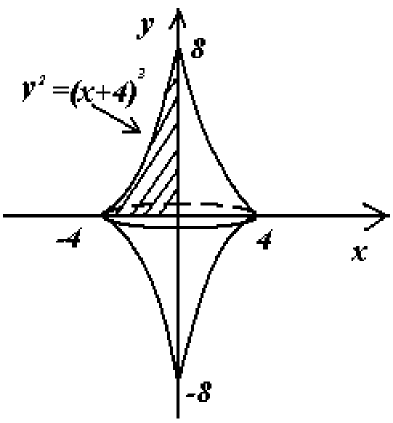
Yechish. (44.1) formulaga binoan:

$$V_x=\pi\int_1^4\left(\frac{4}{x}\right)^2dx=16\pi\int_1^4\left(\frac{1}{x^2}\right)dx=16\pi\left(-\frac{1}{x}\right)\Big|_1^4=16\pi\left(-\frac{1}{4}+1\right)=16\pi\cdot\frac{3}{4}=12\pi.$$

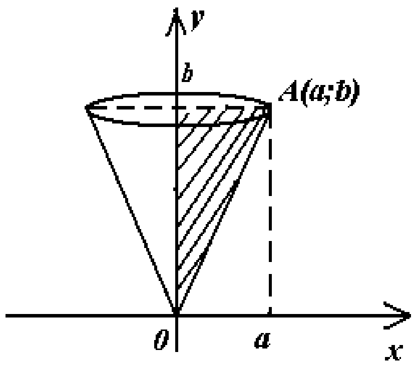
(44.2) formulaga binoan:

$$V_y=2\pi\int_1^4\frac{4}{x}\cdot xdx=8\pi\int_1^4dx=8\pi\cdot x\Big|_1^4=8\pi(4-1)=24\pi$$

2-misol. *$y^2=(x+4)^3$* va *x=0* chiziqlar bilan chegaralangan figura *0y* o'q atorofida aylanishdan hosil bo'lgan jismning hajmini toping (174-chizma).



174-chizma.



175-chizma.

Yechish. Hajmni (44.2) formuladan foydalanib hisoblaymiz. bunda *x<0* ekanini nazarda to'tamiz.

$$V_y=-2\cdot2\pi\int_{-4}^0xydx=-4\pi\int_{-4}^0x(x+4)^{\frac{3}{2}}dx=-4\pi\int_{-4}^0x(x+4-4)(x+4)^{\frac{3}{2}}dx=$$

$$=-4\pi\int_{-4}^0(x+4)^{\frac{5}{2}}dx+16\pi\int_{-4}^0(x+4)^{\frac{3}{2}}dx=-4\pi\frac{(x+4)^{\frac{7}{2}}}{7/2}\Big|_{-4}^0+16\pi\frac{(x+4)^{\frac{5}{2}}}{5/2}\Big|_{-4}^0=$$

$$=-\frac{8\pi}{7}\cdot4^{\frac{7}{2}}+\frac{32\pi}{5}\cdot4^{\frac{5}{2}}=-\frac{8\pi}{7}\cdot2^7+\frac{32\pi}{5}\cdot2^5=\frac{1024\pi}{5}-\frac{1024\pi}{7}=\frac{2048\pi}{35}.$$

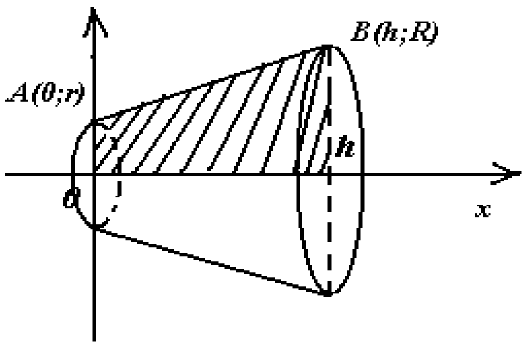
3-misol. Koordinatalar boshini *A(a;b)(a>0, b>0)* nuqta bilan tutashtiruvchi to'g'ri chiziq kesmasi, *oy* o'q hamda *y=b* to'g'ri chiziq bilan chegaralangan figura *oy* o'q atrofida aylanadi. Hosil bo'lgan konusning hajmini hisoblang (175-chizma).

Yechish. *O(0;0)* va *A(a;b)* nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi $x=\frac{a}{b}y$ bo'lishi ravshan. (44.3) formulaga binoan

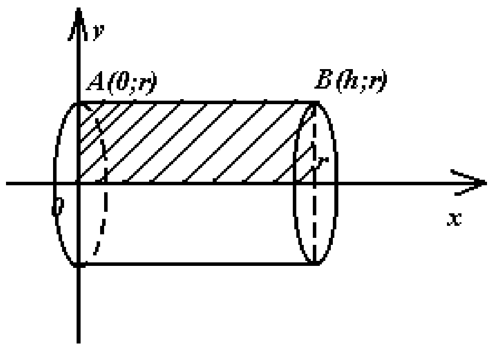
$$V_y=\pi\int_a^b x^2dy=\pi\int_0^b\frac{a^2}{b^2}y^2dy=\frac{\pi a^2}{b^2}\cdot\frac{y^3}{3}\Big|_0^b=\frac{1}{3}\pi a^2 b,$$

ya'ni konusning hajmi asosining yuzi πa^2 bilan balandligi *b* ning ko'paytmasini uchdan biriga teng bo'lar ekan. Bu formula maktab kursidan bizga ayon.

4-misol. *A(0;r)* va *B(h;R)* nuqtalar berilgan, bunda *h>0, R≥r>0. AB* kesma *0x* o'q hamda *x=0, x=h* vertikal to'g'ri chiziqlar bilan chegaralangan figura *0x* o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmini hisoblang.



176-chizma.



177-chizma.

Yechish. *a) AB* kesma *0x* o'qqa parallel bo'lmasin, ya'ni *r≠R*. U holda *AB* kesma *0x* o'q atrofida aylanganda kesik konus hosil bo'ladi (176-chizma).

Ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}.$$

dan floydalanib *A(0;r)* va *B(h;R*) nuqtalrdan o'tuvchi to'g'ri chiziq tenglamasini topamiz:

$$\frac{x-0}{h-0}=\frac{y-r}{R-r} \qquad \text{yoki} \qquad y-r=\frac{R-r}{h}x,\quad y=r+\frac{R-r}{h}x.$$

(44.1) formulaga binoan quyidagini hosil qilamiz:

$$V_x=\pi\int_a^b y^2dx=\pi\int_0^h(r+\frac{R-r}{h}x)^2dx=\pi\int_0^h(r^2+2r\cdot\frac{R-r}{h}x+\left(\frac{R-r}{h}\right)^2x^2)dx=$$

$$.=\pi[r^2x+r\cdot\frac{R-r}{h}x^2+\frac{(R-r)^2}{h^2}\cdot\frac{x^3}{3}]\Big|_0^h=\pi[r^2h+r\cdot(R-r)h+\frac{(R-r)^2}{3}h]=$$

$$=\frac{\pi h}{3}(3r^2+3rR-3r^2+R^2-2rR+r^2)=\frac{\pi h}{3}(R^2+rR+r^2).$$

Shunday qilib asoslarining radiuslari *r*, *R* va balandligi *h* bo'lgan kesik konusning hajmi

$$\vartheta=\frac{\pi h}{3}(R^2+rR+r^2) \quad (44.5)$$

formula yordamida topilar ekan.

b) AB kesma *0x* o'qqa parallel ya'ni *r=R* bo'lsin (177-chizma). U holda *AB* kesma *0x* o'q atrofida aylanganda doiraviy silindr hosil bo'ladim. Qaralayotgan holda *AB* tenglamasi *y=r* ko'rinishga ega bo'ladi. (44.1) formulaga binoan

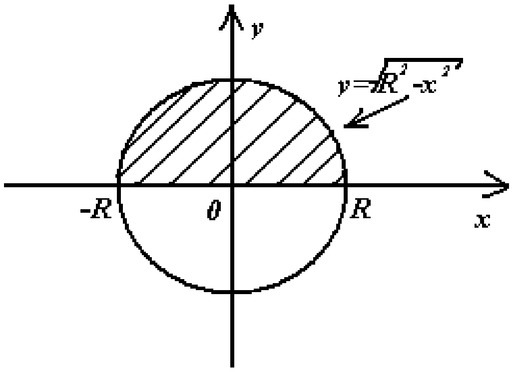
$$\vartheta_x = \pi\int_a^b y^2 dy = \pi\int_0^b r^2 dx = \pi r^2 h$$

bo'ladi. Demak asosining radiusi *r*, balandligi *h* bo'lgan doiraviy silindrning hajmi

$$.\vartheta = \pi r^2 h$$

formula yordamida topilar ekan. Ya'ni to'g'ri doiraviy silindrning hajmi asosining yuzi bilan balandligining ko'paytmasiga teng ekan.

5-misol. $y=\sqrt{R^2-x^2}$ yarim aylana hamda *0x* o'q bilan chegaralangan figura *0x* o'q atrofida aylanadi. Hosil bo'lgan jismning hajmini hisoblang (178-chizma).



178-chizma.

Yechish. (44.1) formulaga binoan:

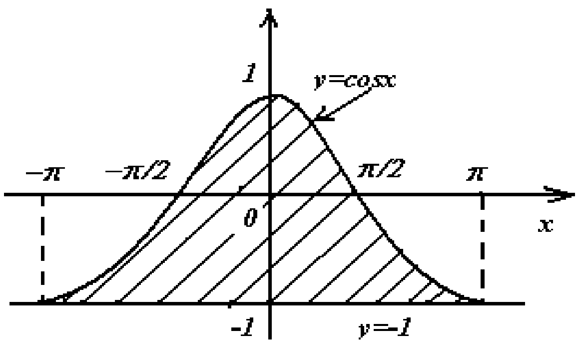
$$\vartheta_x = \pi\int_{-R}^{R}(R^2-x^2)dx = \pi(R^2x-\frac{x^3}{3})\Big|_{-R}^{R} = \pi(R^2\cdot R-\frac{R^3}{3})-\pi(-R^2+\frac{R^3}{3}) = \frac{4}{3}\pi R^3.$$

Agar biz $y=\sqrt{R^2-x^2}$ egri chiziq *0x* o'q bilan chegaralangan soha *0x* o'q atrofida aylanishi natijasida radiusi *R* ga teng shar hosil bo'lishini hisobga olsak sharning hajmini topish uchun

$$\vartheta = \frac{4}{3}\pi R^3$$

formulaga ega bo'lamiz.

6-misol. *y=cosx, y=-1* chiziqlar bilan chegaralangan figuraning *y=-1* to'g'ri chiziq atrofida aylanishidan hosil bo'lgan jismning hajmini toping $(-\pi\le x\le\pi)$ (179-chizma).



179-chizma.

Yechish. $y_1=y+1$, $x_1=x$ almashtirish olsak koordinatalar boshi $0_1(0;-1)$ nuqtada bo'lgan ya'ni $0_1\ x_1y$ sistemaga ega bo'lamiz. Berilgan figura yangi sistemada yuqoridan $y=1+\cos x$ egri chiziq, quyidan $0_1\, x_1$ o'q bilan chegaralangan figuraga aylanadi.

Shuning uchun (44.1) formulaga binoan quyidagini hosil qilamiz:

$$\vartheta_x = \pi\int_a^b y^2dx = \pi\int_{-\pi}^{\pi}(1+\cos x)^2 dx = \pi\int_{-\pi}^{\pi}(1+2\cos x+\cos^2 x)^2 dx =$$

$$= \pi\int_{-\pi}^{\pi}(1+2\cos x+\frac{1+\cos 2x}{2})dx = \pi\left[x+2\sin x+\frac{x+\frac{1}{2}\sin 2x}{2}\right]\Big|_{-\pi}^{\pi} =$$

$$= \pi\left(\frac{3}{2}x+2\sin x+\frac{1}{4}\sin 2x\right)\Big|_{-\pi}^{\pi} = \pi\frac{3}{2}\cdot 2\pi = 3\pi^2.$$

7-misol. *x=a(t-sint), y=a(1-cost)* sikloidaning bir arkasi va *0x* bilan chegaralangan: *a) 0x* o'q; *b) 0y* o'q; *d)* figuraning simmetriya o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping (166-chizma).

Yechish. *a)* (44.1)formulaga binoan: $\vartheta_x = \pi\int_0^{2a\pi} y^2dx.$

t parametrga o'tish uchun *x=a(t-sint), y=a(1-cost), dx=a (1-cost)dt* formulalardan foydalanamiz; *x=0* da *t=0 ; x=2πa* da *t=2π.*

Demak

$$V_x = \pi\int_0^{2\pi} a^2(1-\cos t)^2 a(1-\cos t)dt = \pi a^3\int_0^{2\pi}(1-\cos t)^3 dt = .$$

$$= \pi a^3\int_0^{2\pi}(1-3\cos t+3\cos^2 t-\cos^3 t)dt = \pi a^3\int_0^{2\pi}(1-3\cos t+\frac{3}{2}(1+\cos 2t)-(1-\sin^2 t))\cos t)dt = .$$

$$= \pi a^3(t-3\sin t+\frac{3}{2}t+\frac{3}{2}\cdot\frac{1}{2}\sin 2t-\sin t+\frac{1}{3}\sin^3 t)\Big|_0^{2\pi} = \pi a^3(2\pi+\frac{3}{2}\cdot 2\pi) = 5\pi^2a^3.$$

b) 0y o'q atrofida aylanishdan hosil bo'lgan jismning hajmini (44.2) formuldan foydalanib topamiz:

$$V_y = 2\pi\int_0^{2\pi a} xydx = .2\pi\int_0^{2\pi} a(t-\sin t)a(1-\cos t)a(1-\cos t)dt = .$$

$$= 2\pi a^3\int_0^{2\pi}(t-\sin t)(1-\cos t)^2 dt = 2\pi a^3\int_0^{2\pi}(t-\sin t)(1-2\cos t+\cos^2 t)dt = .$$

$$= 2\pi a^3\int_0^{2\pi}(t-2t\cos t+t\cos^2 t-\sin t+\sin 2t-\sin t\cdot\cos^3 t)dt.$$

2-,4-,5- va *6-* qo'shiluvchilarning integrallarini nolga tengligini tekshirib ko'rish qiyin emas.

Shuning uchun quyidagi integrallarni hisoblash kifoya:

$$\int_0^{2\pi}(t+t\cos^2 t)dt = \int_0^{2\pi}(t+t\cdot\frac{1+\cos 2t}{2})dt = \int_0^{2\pi}\left(\frac{3}{2}t+\frac{1}{2}t\cos 2t\right)dt = \frac{3}{2}\cdot\frac{t^2}{2}\Big|_0^{2\pi}+\frac{1}{4}\int_0^{2\pi} td\sin 2t =$$

$$= \frac{3}{4})(2\pi)^2+\frac{1}{4}t\cdot\sin 2t\Big|_0^{2\pi}-\frac{1}{4}\int_0^{2\pi}\sin 2tdt = 3\pi^2.$$

Demak, $V_y = 2\pi a^3\cdot 3\pi^2 = 6\pi^3\cdot a^3.$

b) Sikloidaning simmetriya o’qi $x=\pi a$ ko’rinishdagi tenglamaga ega. Agar $x_1=x-\pi a$, $y_1=y$ almashtirish olsak koordinatalar boshi $0_1(\pi a, 0)$ nuqtaga ko’chadi va $0\, y_1$ simmetriya o’qi bilan ustma-ust tushadi va (44.2) formuladan foydalanish imkoni tug’iladi.

$x=2\pi a$ da $x_1=a\pi$, $x=a\pi$ da $x_1=0$ bo’lishi ravshan.

Demak,

$$V_y=2\pi\int_a^b xydx=2\pi\int_0^{a\pi}x_1y_1dx_1=2\pi\int_{a\pi}^{2a\pi}(x-a\pi)yd(x-a\pi)=$$

$$=2\pi\int_{a\pi}^{2a\pi}(x-a\pi)ydx=2\pi\int_{\pi}^{2\pi}[a(t-\sin t)-a\pi]a(1-\cos t)a(1-\cos t)dt=$$

$$=2\pi a^3\int_{\pi}^{2\pi}(t-\sin t-\pi)(1-\cos t)^2\,dt.$$

Qavslarni ochib va tegishli integrallarni hisoblab, uzil-kesil topamiz:

$$V_{y_1}=2\pi a^3\left(\frac{3}{4}\pi^2-\frac{4}{3}\right)=\frac{\pi a^3}{6}(9\pi^2-16).$$

8-misol. $\rho=a(1+\cos\theta)$ kardioida hamda qutb o’qi bilan chegaralangan figuraning qutb o’qi atrofida aylanishi natijasida hosil bo’lgan jismning hajmini hisoblang (159-chizma).

Yechish. Kardiolida qutb o’qiga nisbatan simmetrik va θ qutb burchagi 0 dan π gacha o’zgarganda kardioidaning qutb o’qining yuqorisida joylashgan yarmi chiziladi.

Shuning uchun (44.6) formulaga ko’ra quyidagini hosil qilamiz.

$$V_\rho=\frac{2}{3}\pi a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta d\theta=\frac{2}{3}\pi a^3\int_0^{\pi}(\sin\theta+3\sin\theta\cos\theta+3\cos^2\theta\sin\theta+\cos^3\theta\sin\theta)d\theta=$$

$$=\frac{2}{3}\pi a^3\left[\int_0^{\pi}\sin\theta d\theta+\frac{3}{2}\int_0^{\pi}\sin 2\theta d\theta-3\int_0^{\pi}\cos^2\theta d\cos\theta-\int_0^{\pi}\cos^3\theta d\cos\theta\right]=$$

$$=\frac{2}{3}\pi a^3(-\cos\theta-\frac{3}{2}\cos 2\theta-3\cdot\frac{\cos^3\theta}{3}-\frac{\cos^4\theta}{4})\Big|_0^{\pi}=\frac{2}{3}\pi a^3(2+2)=\frac{8\pi a^3}{3}.$$

44.2. Aylanish jismining sirti

Faraz qilaylik $y=f(x)$ funksiya $[a,b]$ kesmada uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo’lsin.

$y=f(x)$ egri chiziqning $x=a$, $x=в$ vertikal to’g’ri chiziqlar orasidagi AB yoyining $0x$ o’q atrofida aylanishi natijasida hosil bo’lgan jismning yon sirtini topish talab etilsin. $[a,b]$ kesmani

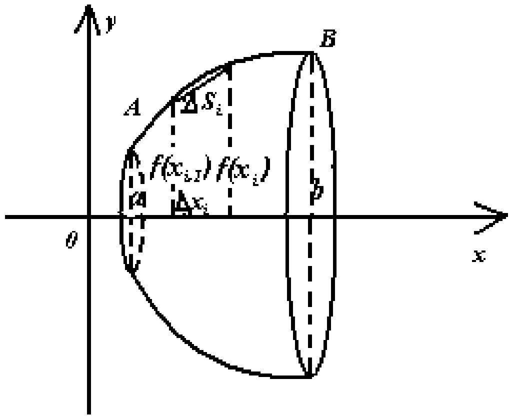
$$a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$$

bo’linish nuqtalari yordamida n ta ixtiyoriy bo’laklarga bo’lamiz. AB egri chiziqda abssissalari $x_0,x_1,x_2,\ldots,x_{n-1},x_n$ nuqtalardan iborat

$A,M_1,M_2,\ldots,M_{i-1},M_i,\ldots,M_{n-1},B$ nuqtalarni olib

AM_1, $M_1M_2,\ldots$, $M_{i-1}M_i$ $,\ldots,M_{n-1}B$ vatarlarni o’tkazamiz va ularning uzunliklarini mos ravishda ΔS_1 , $\Delta S_2,\ldots$, $\Delta S_i,\ldots$, ΔS_n lar orqali belgilaymiz. U holda AB yoyga ichki chizilgan $AM_1M_2... M_{i-1}\ M_i...M_{n-1}B$ siniq chiziqlar hosil bo'ladi. Har qaysi uzunligi

$\Delta S_i\ (i=\overline{1,n})$ bo'lgan $M_{i-1}\ M_i$ vatar $0x$ o'q atrofida aylanishi natijasida kesik konus (yoki silindr) hosil bo'ladi. Bu kesik konusning asoslarining radiuslari $y_{i-1}= f(x_{i-1})$, $y_i= f(x_i)$ bo'lishi ravshan. Kesik konusning yon sirti asos aylanalari uzunliklari yig'indisining yarmi bilan yasovchisi ko'paytmasiga teng edi. Shunga ko'ra *i*-kesik konusning yon sirti



180-chizma.

$$\Delta P_i = 2\pi\frac{f(x_{i-1})+f(x_i)}{2}\Delta S_i$$

bo'ladi. Yoy uzunligini topishda

$$\Delta S_i = \sqrt{1+[f'(z_i)]^2}\Delta x_i, \quad x_{i-1}<z_i<x_i$$

ekanligiga iqror bo'lgan edik. Buni e'tiborga olsak

$$\Delta P_i = \pi[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(z_i)]^2}\Delta x_i$$

bo'ladi.

AB yoyga ichki chizilgan siniq chiziqning $0x$ o'q atrofida aylanishi natijasida hosil bo'lgan jismning yon sirti

$$P_n = \pi\sum_{i=i}^{n}[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(z_i)]^2}\Delta x_i$$

yig'indiga teng bo'ladi. Siniq chiziqning eng katta bo'g'ini (zvenosi)ning uzunligi ΔS_i nolga intilgandagi bu yig'indining limiti qaralayotgan aylanish jismining sirti deyiladi.

$$P = \lim_{\max \Delta S_i\to 0} P_n = \lim_{\max \Delta x_i\to 0} P_n = \lim_{\max \Delta x_i\to 0}\pi\sum_{i=i}^{n}[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(z_i)]^2}\Delta x_i =$$

$$=\pi\lim_{\max \Delta x_i\to 0}\sum_{i=i}^{n}2f(z_i)\sqrt{1+[f'(z_i)]^2}\Delta x_i = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx.$$

Shunday qilib aylanish jismining yon sirti

$$P_x = 2\pi\int_a^b y\sqrt{1+y'^2}dx = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx. \quad (44.7)$$

formula yordamida topilar ekan.

Eslatma. Aylanish jismining to'la sirtini topish talab etilganda uning yon sirtiga asoslarining radiuslari *f(a)* va *f(b)* bo'lgan doiralarning yuzlarini qo'shiladi.

Agar egri chiziq $x=\varphi(t)$, $y=\psi(t)$ $\alpha\leq t\leq\beta$ parametrik tenglamalari yordamida berilgan bo'lib $\varphi(t)$, $\psi(t)$ funksiyalar $[\alpha, \beta]$ kesmada uzluksiz va uzluksiz hosilalarga ega bo'lsa hamda $\varphi(\alpha)=a$, $\psi(\beta)=b$ bo'lsa, u holda (44.7) formulada $x=\varphi(t)$, almashtirish olib aylanish jismining yon sirtini topish uchun

$$P_x = 2\pi\int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt. \quad (44.8)$$

formulani hosil qilamiz.

Egri chiziq qutb koordinatalar sistemasida $\rho=f(\theta)$, $\alpha\leq\theta\leq\beta$ tenglama bilan berilgan bo'lib egri chiziqli sektorni ρ qutb o'qi atrofida aylanishi natijasida hosil bo'lgan jismning sirti yuzini topish talab etilsin. Agar $f(\theta)$ funksiya $[\alpha, \beta]$ kesmada uzluksiz va uzluksiz $f'(\theta)$ hosilaga ega bo'lsa egri chiziqni $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$ parametrik tenglamalari yordamida berilgan deb qarash mumkin. Shuning uchun bu holda aylanish jismining yon sirti

$$P_\rho = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{f^2(\theta)+f'^2(\theta)}d\theta.$$

formula yordamida topiladi.

9-misol. $y=\sqrt{R^2-x^2}$ yarim aylanani *0x* o'q atrofida aylanishi natijasida hosil bo'lgan sferaning sirtini hisoblang.

Yechish.

$$y' = (\sqrt{R^2-x^2})' = -\frac{x}{\sqrt{R^2-x^2}}; \qquad \sqrt{1+y'^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}}.$$

$$P_x = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\cdot\frac{R}{\sqrt{R^2-x^2}}dx = 2\pi Rx\Big|_{-R}^{R} = 4\pi R^2.$$

10-misol. *x=a(t-sint), y=a(1-cost),* $0\leq t\leq 2\pi$ sikloida bitta arkasini *0x* o'q atrofida aylanishi natijasida hosil bo'lgan jismning sirtini hisoblang.

Yechish. $\sqrt{x'^2+y'^2} = 2a\sin\frac{t}{2}$ edi (43-ma'ruza 4-misolga qarang).

Shuning uchun (44.8) formulaga binoan quyidagiga ega bo'lamiz.

$$P_x = 2\pi\int_0^{2\pi} a(1-\cos t)2a\sin\frac{t}{2}dt = .8\pi a^2\int_0^{2\pi}\sin^2\frac{t}{2}\cdot\sin\frac{t}{2}dt = .$$

$$= 8\pi a^2\int_0^{2\pi}(1-\cos^2\frac{t}{2}(-2)d(\cos\frac{t}{2})dt = -16\pi a^2\left[\cos\frac{t}{2}-\frac{\cos^3\frac{t}{2}}{3}\right]\Big|_0^{2\pi} = -16\pi a^2(-2+\frac{2}{3}) = \frac{64}{3}\pi a^2.$$

11-misol. $\rho = a(1+\cos\theta)$ kardiodaning qutb o'qi atrofida aylanishidan hosil bo'lgan jismning sirtning yuzini toping (159-chizma).

Yechish. (44.9) formulaga binoan:

$$P_p = 2\pi\int_0^{\pi}\rho\sin\theta\sqrt{\rho^2+\rho'^2}\,d\theta = 2\pi\int_0^{\pi}a(1+\cos\theta)\sin\theta\sqrt{[a(1+\cos\theta)]^2+(a\sin\theta)^2}\,d\theta =$$

$$= 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta\cdot 2\cdot\cos\frac{\theta}{2}\,d\theta = 4\pi a^2\int_0^{\pi}2\cos^2\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\,d\theta =$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\,d\theta = 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}(-2)d(\cos\frac{\theta}{2}). = -32\pi a^2\,\frac{\cos^5\frac{\theta}{2}}{5}\Big|_0^{\pi} = \frac{32\pi a^2}{5}.$$

O'z-o'zini tekshirish uchun savollar

1. *y=f(x), y=0, x=a, x=b* chiziqlar bilan chegaralangan figuraning *0x* va *0y* o'qlar atrofida aylanishi natijasida hosil bo'lgan jismlarning hajmi qanday topiladi?
2. *x=φ(y), x=0, y=c, y=d (c<d)* chiziqlar bilan chegaralangan figuraning *0x* o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmi qanday topiladi?
3. $y=f_1(x)$, $y=f_2(x)$ $(f_1(x) \le f_2(x))$ *x=a, x=b (a<b)* chiziqlar bilan chegaralangan figurani *0x* hamda *0y* o'qlari atrofida aylanishi natijasida hosil bo'lgan jismlarning hajmlari qanday topiladi?
4. Qutb koordinatalar sistemasida berilgan egri chiziqli sektorni qutb o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi qanday topiladi?
5. *y=f(x)* egri chiziqni *AB* yoyini *0x* o'q atrofida aylanishi natijasida hosil bo'lgan jismni yon sirti qanday topiladi?
6. Egri chiziq parametrik tenglamalari yordamida berilganda aylanish jismining yon sirti qanday topiladi?
7. Qutb koordinatalar sistemasida berilgan egri chiziqli sektorni qutb o'qi atrofida aylanishidan hosil bo'lgan jismning yon sirti qanday topiladi?
8. Shar sirtini topish formulasini keltirib chiqaring.
9. Konusning hajmini topish formulasini keltirib chiqaring.
10. Konusning yon sirtini topish formulasini keltirib chiqaring.
11. Kesik konusning hajmi va yon sirtini topish formulalarini keltirib chiqaring.
12. Silindrning hajmi va yon sirtini topish formulalarini keltirib chiqaring.

Mustaqil yechish uchun mashqlar

1. $y=x^2+3$, *x=0, x=4, y=0* chiziqlar bilan chegaralangan figuraning *0x* o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: *368,8 π.*

2. $y=2x-x^2$ va *0x* o'q bilan chegaralangan figuraning *0x* o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $\frac{16}{15}\pi$.

3. $xy=9$, $y=3$, $y=9$, $x=0$ chiziqlar bilan chegaralangan figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: 18π.

4. $y=10-x^2$ va $y=x^2+2$ parabolalar hamda $0y$ o'q bilan chegaralangan figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmi topilsin. Javob: 16π

5. $y=4-x^2$ parabola hamda $2x+y-4=0$ to'g'ri chiziq bilan chegaralangan figuraning $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $\frac{32\pi}{5}$.

6. $x=a\cos^3 t$, $y=\sin^3 t$ astroidaning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping (155-chizma) Javob: $\frac{32\pi a^3}{105}$.

7. $\rho=a\cos^2\theta$ egri chiziq hamda qutb o'qi bilan chegaralangan figurani qutb o'qi atrofidan aylanishidan hosil bo'lgan jismning hajmini topilsin. Javob: $\frac{4}{21}\pi a^3$.

8. $x^2=4+y$ egri chiziq va $y=-2$ to'g'ri chiziq bilan chegaralangan figurani $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning sirtini hisoblang. Javob: $\frac{34\sqrt{17}-2}{9}\pi$.

9. $\rho=2a\sin\theta$ aylanani qutb o'qi atrofida aylanishidan hosil bo'lgan jismning sirtini toping. Javob: $4\pi^2a^2$

10. $x=a\cos^3 t$, $y=\sin^3 t$ astroidaning abssissalar o'qi atrofida aylanishidan hosil bo'lgan jismning sirtini toping. Javob: $\frac{6\pi a^2}{5}$.

45-ma'ruza. Mavzu: Aniq integralning mexaniqaga tatbiqlari

Reja:

1. Statik moment.
2. Inersiya momenti.
3. Tekislikdagi chiziqning og'irlik markazi va statik hamda inersiya momentlari.
4. Yassi (tekis) figura og'irlik markazi va statik hamda inersiya momentlari.

Adabiyotlar: 1,2,3,4,5,6,7,9,12.

Tayanch iboralar: statik moment, inersiya momenti, og'irlik markazi.

45.1.Statik moment

m massali moddiy nuqtaning ℓ o'qqa nisbatan **statik momenti** deb $M_e=md$ kattalikka aytiladi, bu yerda d nuqtadan ℓ o'qqacha bo'lgan masofa.

Agar oxy tekislikda massalari $m_1, m_2, \ldots, m_i, \ldots m_n$ bo'lgan moddiy nuqtalarning

$$P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_i(x_i, y_i), \ldots, P_n(x_n, y_n) \qquad (45.1)$$

sistemasi berilgan bo'lsa , u holda x_i m_i va y_i m_i ko'paytmalar m_i massaning *oy* va *ox* o'qlarga nisbatan **statik momentlari** deyiladi. Berilgan (45.1) moddiy nuqtalar sistemasining og'irlik markazi koordinatalari

$$x_c = \frac{x_1 m_1 + x_2 m_2 + \ldots + x_n m_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \quad (45.2)$$

$$y_c = \frac{y_1 m_1 + y_2 m_2 + \ldots + y_n m_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i} \quad (45.3)$$

formulalar yordamida topilishi mexanika kursidan ma‘lum.

$\sum_{i=1}^{n} x_i m_i$ va $\sum_{i=1}^{n} y_i m_i$ yigindilar berilgan sistemaning mos ravishda *oy* va *ox* o'qlarga nisbatan **statik momenti** deyiladi, ya‘ni

$$M_x=, \sum_{i=1}^{n} y_i m_i \quad M_y=\sum_{i=1}^{n} x_i m_i \quad (45.4)$$

Agar moddiy nuqtalar sistemasi *ox* yoki *oy* o'qqa nisbatan simmetrik bo'lsa, u holda uning mos statik momentlari nolga teng bo'ladi.

45.2. Inersiya momenti

m massali moddiy nuqtaning ℓ o'qqa nisbatan inersiya momenti deb $I_l=md^2$ songa aytiladi, bu yerda *d*- nuqtadan o'qqacha masofa.

(45.1) moddiy nuqtalar sistemasining *ox* va *oy* o'qqa nisbatan inersiya momentlari

$$I_x = \sum_{i=1}^{n} y_i^2 m_i, \quad I_y = \sum_{i=1}^{n} x_i^2 m_i \quad (45.5)$$

fomulalar yordamitda topiladi.

(45.2), (45.3),(45.4) va (45.5) formulalardan geometrik figura va jismlarning og'irlik markazini, statik va inersiya momentlarini topishda foydalanamiz.

45.3. Tekislikdagi chiziqning og'irlik markazi va statik hamda inersiya momentlari

Uzluksiz *AB* egri chiziq $y=f(x)$, $a \le x \le b$ tenglama bilan berilgan bo'lsin.

Berilgan chiziq uzunlik birligining massasi **chiziqli zichlik** deyiladi. Egri chiziqning hamma joyida chiziqli zichlik bir xil va γ ga teng (egri chiziq bir jinsli) deb faraz qilamiz.

AB egri chiziqni uzunliklari $\Delta s_1, \Delta s_2, \ldots \Delta s_n$, bo'lgan ixtiyoriy *n* ta bo'lakka bo'lamiz. Bu bo'laklarning massalari ularning uzunliklari bilan chiziqli zichlik ko'paytmasiga teng, ya‘ni Δs_i $(i=\overline{1,n})$ ning massasi $\Delta m_i = \gamma \cdot \Delta s_i$ bo'ladi. *AB* yoyining har bir Δs_i bo'lagida abssissasi z_i bo'lgan $P_i\,(z_i, f(z_i))$ nuqta olamiz. *AB* yoyning har bir Δs_i bo'lagini massasi $\gamma \Delta s_i$ bo'lgan $P_i\,(z_i, f(z_i))$ moddiy nuqta deb qarab (45.2) va (45.3) formulalarda x_i o'rniga z_i ,y_i o'rniga $f(z_i)$,m_i o'rniga Δ

$m_i=\gamma \Delta s_i$ qiymatlarni qo'ysak, *AB* yoyining og'irlik markazini aniqlash uchun quyidagi taqtribiy formulalarni hosil qilamiz:

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \gamma \Delta s_i}{\sum_{i=1}^{n} \gamma \Delta s_i}, y_c \approx \frac{\sum_{i=1}^{n} f(z_i) \gamma \Delta s_i}{\sum_{i=1}^{n} \gamma \Delta s_i}$$

y=f(x) [a,b] kesmada uzluksiz va uzluksiz *f ′(x)* hosilaga ega bo'lsin. U holda $\Delta s_i = \sqrt{1+[f(z_i)]^2}\Delta x_i$ ekanligi ko'rsatilgan edi. Shuning uchun

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \sqrt{1+[f'(z_i)]^2}\Delta x_i}{\sum_{i=1}^{n} \sqrt{1+[f'(z_i)]^2}\Delta x_i}, y_c \approx \frac{\sum_{i=1}^{n} f(z_i) \sqrt{1+[f'(z_i)]^2}\Delta x_i}{\sum_{i=1}^{n} \sqrt{1+[f'(z_i)]^2}\Delta x_i}$$

ga ega bo'lamiz.

Bu formulalarda $max\Delta x_i \to o$ da limitga o'tib, *AB* yoyning og'irlik markazi koordinatalarini topish uchun quyidagi aniq formulalarni hosil qilamiz:

$$x_c = \frac{\int_a^b x\sqrt{1+y'^2}\,dx}{\int_a^b \sqrt{1+y'^2}\,dx} \quad (45.2'), \quad y_c = \frac{\int_a^b y\sqrt{1+y'^2}\,dx}{\int_a^b \sqrt{1+y'^2}\,dx} \ . \quad (45.3')$$

Bu formulalardan *AB* yoyni statik momentini hisoblash uchun

$$M_x = \int_a^b y\sqrt{1+y'^2}\,dx, \ M_y = \int_a^b x\sqrt{1+y'^2}\,dx \qquad (45.4')$$

formulalarga ega bo'lamiz.

AB yoyning inersiya momentlarini hisoblash uchun quyidagi formulalarni ham xuddi shu usulda hosil qilish mumkin:

$$I_x = \int_a^b y^2\sqrt{1+y'^2}\,dx, \ I_y = \int_a^b x^2\sqrt{1+y'^2}\,dx \qquad (45.5`)$$

(45.2′) va (45.3′) formulalarning maxrajidagi ifoda *AB* yoyning massasi *M* ni ifodalashini ta'kidlab o'tamiz.

AB egri chiziq *x=φ(t), y=ψ(t), α≤t≤β* tenglamalar yordamida berilgan bo'lib *φ(t), ψ(t)* funksiyalar *[α,β]* kesmada uzluksiz va uzluksiz hosilalarga ega hamda *φ(α)=a* , *φ(β)=b* bo'lsa, u holda yuqorida chiqarilgan formulalarda *x= φ(t)* almashtirish olib quyidagilarni hosil qilamiz:

$$x_c = \frac{\int_\alpha^\beta \varphi(t)\sqrt{[\varphi'(t)]^2+[\psi' t)]^2}\,dt}{\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\psi' t)]^2}\,dt} \quad (45.2''), \quad y_c = \frac{\int_\alpha^\beta \psi(t)\sqrt{[\varphi'(t)]^2+[\varphi'(t)]^2}\,dt}{\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\varphi'(t)]^2}\,dt} \quad (45.3'')$$

$$M_x=\int_\alpha^\beta \psi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt,\;\; M_y=\int_\alpha^\beta \varphi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \qquad (45.4'')$$

$$I_x=\int_\alpha^\beta \psi^2(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt,\;\; I_y=\int_\alpha^\beta \varphi^2(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \qquad (45.5``)$$

AB egri chiziq qutb koordinatalar sistemasida $\rho=f(\theta)$, $\alpha\le\theta\le\beta$ tenglama bilan berilgan bo'lib $f(\theta)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz $f'(\theta)$ hosilaga ega bo'lsin. U holda *AB* yoy og'irlik markazining abssissasi va ordinatasi

$$x_c=\frac{\int_\alpha^\beta \rho\cos\theta\sqrt{\rho^2+\rho'^2}\,d\theta}{\int_\alpha^\beta \sqrt{\rho^2+\rho'^2}\,d\theta} \quad (45.2'''),\qquad y_c=\frac{\int_\alpha^\beta \rho\sin\theta\sqrt{\rho^2+\rho'^2}\,d\theta}{\int_\alpha^\beta \sqrt{\rho^2+\rho'^2}\,d\theta} \quad (45.3''')$$

formulalar yordamida topiladi.

AB yoyni qutb o'qiga nisbatan statik momenti

$$M_\rho=M_x=\int_\alpha^\beta \rho\sin\theta\sqrt{\rho^2+\rho'^2}\,d\theta \qquad (45.4''')$$

formula yordamida shu yoyning qutb o'qiga nisbatan inersiya momenti

$$I_\rho=I_x=\int_\alpha^\beta \rho^2\sin^2\theta\sqrt{\rho^2+\rho'^2}\,d\theta \qquad (45.5''')$$

formula yordamida topiladi.

1-misol. *ox* o'qning yuqorisida joylashgan $x^2+y^2=a^2$ yarim aylana og'irlik markazinining koordinatalari topilsin.

Yechish.

$$y=\sqrt{a^2-x^2},\quad y'=-\frac{x}{\sqrt{a^2-x^2}},\quad \sqrt{1+y'^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}$$

(45.3`) formulaga binoan

$$y_c=\frac{\int_{-a}^{a}\sqrt{a^2+x^2}\cdot\frac{a}{\sqrt{a^2-x^2}}dx}{a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}}=\frac{ax\Big|_{-a}^{a}}{a\arcsin\frac{x}{a}\Big|_{-a}^{a}}=\frac{2a^2}{2a\arcsin 1}=\frac{a}{\frac{\pi}{2}}=\frac{2a}{\pi}.$$

Qaralayotgan yarim aylana *oy* o'qqa nisbatan simmetrik bo'lganligi uchun $x_c=0$ bo'ladi.

Demak, $c(o;\ \frac{2a}{\pi})$ nuqta berilgan yarim aylananing og'irlik markazidir.

2-misol. $x=a\cos^3 t$, $y=a\sin^3 t$ astroidaning birinchi kvadrat (chorak) da yotgan yoyining *ox* va *oy* o'qlarga nisbatan statik momentlarini va og'irlik markazini toping (155-chizma).

Yechish. $x'=-3a\cos^2 t\sin t$, $y'=3a\sin^2 t\cos t$,

$$\sqrt{x'^2+y'^2}=\sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}=\sqrt{9a^2\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)}=3a\cos t\sin t.$$

(45.4″) formulaga asosan:

$$M_x=\int_0^{\frac{\pi}{2}} a\sin^3 t\cdot 3a\cos t\sin t dt=3a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos t dt=3a^2\int_0^{\frac{\pi}{2}}\sin^4 t d(\cos t)=3a^2\frac{\sin^5 t}{5}\Big|_0^{\frac{\pi}{2}}=\frac{3a^2}{5},$$

$$M_y=\int_0^{\frac{\pi}{2}} a\cos^3 t\cdot 3a\cos t\sin t dt=-3a^2\int_0^{\frac{\pi}{2}}\cos^4 t d\cos t=-\frac{3a^2}{5}\cos^5 t\Big|_0^{\frac{\pi}{2}}=\frac{3a^2}{5}.$$

Massani topamiz:

$$M=\int_0^{\frac{\pi}{2}}\sqrt{x'^2+y'^2}dt=3a\int_0^{\frac{\pi}{2}}\cos t\sin t dt=3a\int_0^{\frac{\pi}{2}}\sin t d(\sin t)=3a\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}=\frac{3a}{2}.$$

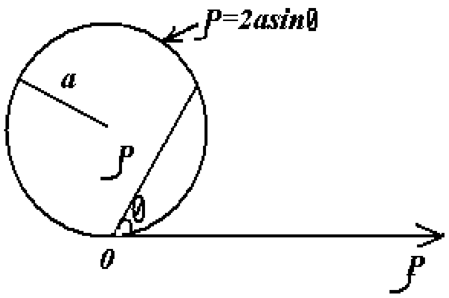
Demak,

$$x_c=\frac{M_y}{M}=\frac{3a^2}{5}:\frac{3a}{2}=\frac{2}{5}a=0{,}4a,$$

$$y_c=\frac{M_x}{M}=0{,}4a.$$

Shunday qilib $M_x=M_y=0{,}6a^2$, $c\ (0{,}4a;\ 0{,}4a)$.

3-misol. $\rho=2a\sin\theta$ aylananing qutb o'qiga nisbatan statik momentini toping (181-chizma).



181-chizma.

Yechish. Qutb burchagi θ 0 dan π gacha o'zgarganda aylana chiziladi. Shuning uchun (45.4```) formulaga binoan quyidagiga ega bo'lamiz.

$$M_\rho=M_x=\int_o^{\pi}2a\sin\theta\sin\theta\sqrt{(2a\sin\theta)^2+(2a\cos\theta)^2}d\theta=4a^2\int_o^{\pi}\sin^2\theta d\theta=4a^2\int_o^{\pi}\frac{1-\cos 2\theta}{2}d\theta=$$

$$=2a^2(\theta-\frac{1}{2}\sin 2\theta)\Big|_0^{\pi}==2a^2\pi.$$

45.4. Yassi (tekis) figura og'irlik markazi va statik hamda inersiya momentlari

Berilgan figura uzluksiz $y=f_1(x)$, $y=f_2(x)$, $(f_1(x)\leq f_2(x))$ egri chiziq hamda $x=a$, $x=b$ $(x<b)$ to'g'ri chiziqlar bilan chegaralangan bo'lsin. Uning sirt zichligi, ya'ni yuz birligiga mos massa hamma joyda bir xil va δ ga teng deb faraz qilamiz

$(\delta = const)$. Bunaqa figura odatda bir jinsli deb yuritiladi. Shu figuraning og'irlik markazini topamiz. (182-chizma).

Berilgan figurani $x=a=x_o, x=x_1, x=x_2,\dots x=x_n=b$ to'g'ri chiziqlar bilan kengligi $\Delta x_1, \Delta x_2,\dots,\Delta x_n$ bo'lgan n ta bo'laklarga ajratamiz.

Har bir bo'lakning massasi uning yuzi bilan δ chiziqli zichlik ko'paytmasiga teng bo'ladi.

Agar i-bo'lakni asosi Δx_i va balandligi $f_2(z_i)$- $f_1(z_i)$ bo'lgan to'g'ri to'rtburchak bilan almashtirsak, i-bo'lakning massasi

$$\Delta m_i = \delta[f_2(z_i) - f_1(z_i)]\,\Delta x_i$$

ga teng bo'ladi, bunda $z_i = \frac{x_{i-1} + x_i}{2}$ (ya'ni z_i $[x_{i-1}, x_i]$ kesmaning o'rtasi). Bu bo'lakning og'irlik markazi taxminan tegishli to'g'ri to'rtburchakning markazi(diogonallarining kesishish nuqtasi) da, ya'ni koordinatalari

$$(x_i)_c = z_i, \qquad (y_i)_c = \frac{f(z_i) + f_i(z_i)}{2}$$

bo'lgan nuqtada bo'ladi.

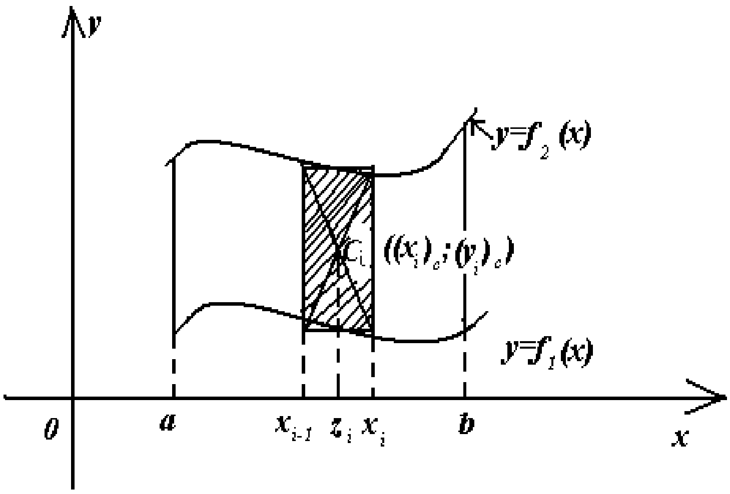
Endi har bir bo'lakni massasi tegishli to'g'ri to'rtburchakning massasiga teng bo'lgan va shu to'g'ri to'rtburchak og'irlik markaziga to'plangan moddiy nuqta bilan almashtiramiz.

U holda butun figura og'irlik markazi koordinatalarini topish uchun moddiy nuqtalar sistemasi og'irlik markazi koordinatalarini topish formulalari (45.2) va (45.3) dan foydalanish mumkin. Bu formulalarga x_i o'rniga z_i, y_i o'rniga $\frac{1}{2}[f_2(z_i) + f_1(z_i)]$, m_i o'rniga $\Delta m_i = \delta[f_2(z_i) - f_1(z_i)]\,\Delta x_i$ ni quyib quyidagiga ega bo'lamiz.

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \delta[f_2(z_i) - f_1(z_i)\Delta x_i}{\sum_{i=1}^{n} \delta[f_2(z_i) - f_1(z_i)\Delta x_i}, \qquad y_c \approx \frac{\frac{1}{2}\sum_{i=1}^{n}[f_2(z_i) + f_1(z_i)]\delta[f_2(z_i) - f_1(z_i)]\Delta x_i}{\sum_{i=1}^{n} \delta[f_2(z_i) - f_1(z_i)]\Delta x_i}$$

Bunda $\max \Delta x_i \to o$ da limitga o'tib, berilgan figura og'irlik markazining koordinatalarini topamiz:

$$x_c = \frac{\int_a^b x[f_2(x) - f_1(x)]dx}{\int_a^b [f_2(x) - f_1(x)]dx}, \qquad y_c = \frac{\frac{1}{2}\int_a^b [f_2(x) + f_1(x)][f_2(x) - f_1(x)]dx}{\int_a^b [f_2(x) - f_1(x)]dx} \qquad (45.6)$$



182-chizma.

Bu formulalarning maxrajida berilgan figuraning massasi turishini hamda chiziqli zichlik δ o'zgarmas bo'lganligi tufayli hisoblash jarayonida qisqarib ketganligini qayd etamiz. Berilgan figura bir jinsli bo'lmasa, ya'ni chiziqli zichlik δ o'zgaruvchi bo'lganda figuraning og'irlik markazi koordinatalari chiziqli zichlikka bog'liq bo'ladi.

Shunday qilib (45.6)formulalardan chiziqli zichlik $\delta=1$ deb faraz qilib

$$M_x = \frac{1}{2}\int_a^b [f_2(x)+f_1(x)][f_2(x)-f_1(x)]dx, \qquad (45.7)$$

va

$$M_y = \int_a^b x[f_2(x)-f_1(x)]dx \qquad (45.8)$$

berilgan bir jinsli figuraning *ox* va *oy* o'qlarga nisbatan statik momentlarini topish formulalarini hamda

$$M = \int_a^b [f_2(x)-f_1(x)]dx \qquad (45.9)$$

shu figuraning masasini topish formulasini hosil qilamiz.

Agar berilgan bir jinsli figura (zichlik $\delta=1$) $y=f(x)\geq o,\ y=0,\ x=a,\ x=b\ (a<b)$ chiziqlar bilan chegaralangan bo'lsa, u holda yuqorida keltirilgan formulalar quyidagi ko'rinishni oladi:

$$M_x = \frac{1}{2}\int_a^b y^2dx = \frac{1}{2}\int_a^b f^2(x)dx \quad (45.7'), \quad M_y = \int_a^b xydx = \int_a^b xf(x)dx \quad (45.8'),$$

$$M = \int_a^b ydx = \int_a^b f(x)dx. \quad (45.9')$$

Shu figuraning koordinata o'qlariga nisbatan inersiya momentlari

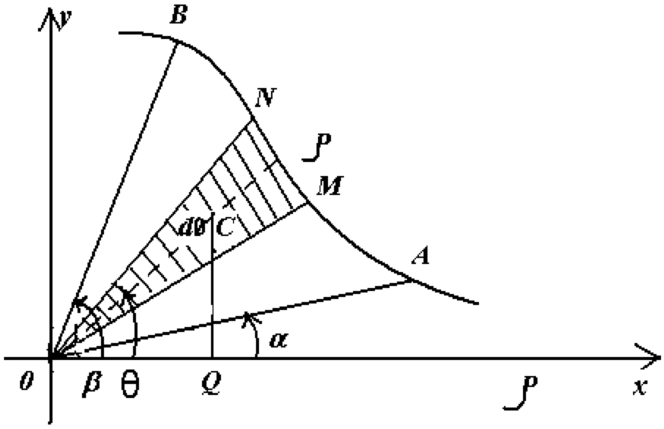
$$I_x = \frac{1}{3}\int_a^b y^3dx = \frac{1}{3}\int_a^b f^3(x)dx \quad (45.10), \quad I_y = \int_a^b x^2ydx = \int_a^b x^2f(x)dx \quad (45.11),$$

formulalar yordamida topilishini ta'kidlab o'tamiz.

Endi ikkita $\theta=\alpha, \quad \theta=\beta \quad (\alpha<\beta)$ nurlar va qutb koordinatalar sistemasida $\rho=\rho(\theta)$ tenglamaga ega uzluksiz egri chiziq bilan chegaralangan bir jinsli sektorning og'irlik markazining abssissasi va ordinatasi (183-chizma)

$$x_c=\frac{2}{3}\frac{\int_\alpha^\beta \rho^3\cos\theta d\theta}{\int_\alpha^\beta \rho^2 d\theta}, \quad y_c=\frac{2}{3}\frac{\int_\alpha^\beta \rho^3\sin\theta d\theta}{\int_\alpha^\beta \rho^2 d\theta}, \quad (45.12)$$

formulalar orqali topilishini isbotlaymiz



183-chizma.

Egri chiziqli *OMN* sektorni qaraylik. Kichik $d\theta$ lar uchun bu sektorning yuzi

$$S_{OMN}=\frac{1}{2}\rho^2 d\theta$$

bo'ladi. Uchburchakning og'irlik markazi uning medianalari kesishgan nuqtada joylashganligi elementar geometriyadan ma'lum. Egri chiziqli *OMN* sektorni og'irlik markazi *C* nuqtada va massasi $\Delta m=\frac{1}{2}\rho^2 d\theta$ uning og'irlik markazida joylashgan uchburchak deb faraz qilib, uning *0x* o'qqa nisbatan statik momentini hisoblaymiz. ΔQCO dan

$$\frac{CQ}{OC}=\sin\theta, \qquad CQ=OC\sin\theta.$$

Uchburchak medianasining xossasiga ko'ra mediana kesishish nuqtasida uning uchidan hisoblanganda *2:1* nisbatda bo'linadi. Shunga ko'ra $OC=\frac{2}{3}\rho$ va $CQ=\frac{2}{3}\rho\sin\theta$ bo'ladi.

Demak, *OMN* sektorning *0x* o'qqa nisbatan statik momenti

$$dM_x=CQ\cdot dm=\frac{2}{3}\rho\sin\theta\cdot\frac{1}{2}\rho^2 d\theta=\frac{1}{3}\rho^3\sin\theta d\theta$$

bo'ladi. Bu tenglikni α dan β gacha integrallab *OAB* sektorning *0x* o'qqa (qutb o'qiga) nisbatan statik momentini hosil qilamiz:

$$M_x=\frac{1}{3}\int_\alpha^\beta \rho^3\sin\theta d\theta. \qquad (45.13)$$

Xuddi shunday

$$M_y = \frac{1}{3}\int_{\alpha}^{\beta} \rho^3 \cos\theta d\theta \qquad (45.14)$$

ekanini topamiz.

Ma‘lumki, *OAB* sektorning yuzi

$$S = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2 d\theta$$

formula yordamida topiladi. Figuraning yuzi uning massasiga teng ekanligini nazarda to’tib (geometrik figuralarning chiziqli zichligi birga teng deb olinadi)

$$x_c = \frac{M_y}{M} = \frac{M_y}{S}, \quad y_c = \frac{M_x}{M} = \frac{M_x}{S}$$

tengliklarga M_y, M_x va *S* o’rniga ularning topilgan qiymatlarini qo’ysak isbotlanishi lozim bo’lgan (45.12)formulalar hosil bo’ladi.

4-misol. *y=sinx* sinusoida yoyi va *0x* o’q bilan chegaralangan figura ($o \le x \le \pi$) og’irlik markazini toping.

Yechish. Berilgan figura $x = \frac{\pi}{2}$ to’g’ri chiziqqa nisbatan simmetrik bo’lganligi sababli $x_c = \frac{\pi}{2}$ bo’ladi. (45.7′) formulaga binoan:

$$M_x = \frac{1}{2}\int_0^{\pi} \sin^2 x dx = \frac{1}{2}\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{4}\int_0^{\pi} (1-\cos 2x)dx = \frac{1}{4}(x - \frac{1}{2}\sin 2x)\Big|_o^{\pi} = \frac{1}{4}\cdot\pi = \frac{\pi}{4}.$$

(45.9′) formulaga binoan figuraning massasi

$$M_x = \int_0^{\pi} \sin x dx = -\cos x\Big|_o^{\pi} = 2$$

bo’ladi Demak,

$$y_c = \frac{M_x}{M} = \frac{\pi}{4} : 2 = \frac{\pi}{8}$$

va figuraning og’irlik markazi $C(\frac{\pi}{2}; \frac{\pi}{8})$ ekan.

5-misol. $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$ $o \le t \le 2\pi$ astroidaning birinchi kvadrat (chorak) dagi qismi va koordinata o’qlari bilan chegaralangan figuraning og’irlik markazi topilsin (155-chizma).

Yechish. Simmetriyaga ko’ra $x_c = y_c$. Statik moment M_y ni (45.8′) formuladan foydalanib topamiz

$$M_y = \int_o^a xydx = \int_{\frac{\pi}{2}}^{0} a\cos^3 t a\cdot\sin^3 t(a\cos^3 t)' dt = -a^3\int_{\frac{\pi}{2}}^{0} \cos^3 t\cdot\sin^3 t\cos^2 t\sin t dt = 3a^2\int_0^{\frac{\pi}{2}} \cos^5 t\cdot\sin^4 t dt =$$

$$= 3a^2\int\limits_0^{\frac{\pi}{2}}\cos^4 t\cdot\sin^4 t\cos t dt = 3a^2\int\limits_0^{\frac{\pi}{2}}(1-\sin^2 t)^2\cdot\sin^4 t d\sin t = 3a^2\int\limits_0^{\frac{\pi}{2}}(\sin^4 t-2\sin^6 t+\sin^8 t)d\sin t =$$

$$= 3a^2\left(\frac{\sin^5 t}{5}-2\cdot\frac{\sin^7 t}{7}+\frac{\sin^9 t}{9}\right)\Bigg|_0^{\frac{\pi}{2}} = 3a^2\left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)=\frac{8a^3}{105}.$$

Figura massasi $(45.9')$ ga ko'ra

$$M = S = \int\limits_o^a ydx = \int\limits_{\frac{\pi}{2}}^0 a\sin^3 t(a\cos^3 t)'dt = -a^2\int\limits_{\frac{\pi}{2}}^0 \sin^3 t 3\cos^2 t\sin t dt = 3a^2\int\limits_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t dt =$$

$$= 3a^2\int\limits_0^{\frac{\pi}{2}}(\sin^2 t)^2\cos^2 t dt = 3a^2\int\limits_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2t}{2}\right)^2\cdot\frac{1+\cos 2t}{2}dt = \frac{3a^2}{8}\int\limits_0^{\frac{\pi}{2}}(1-\cos 2t)(1-\cos^2 2t)dt =$$

$$= \frac{3a^2}{8}\int\limits_0^{\frac{\pi}{2}}(1-\cos 2t)\sin^2 2t dt = \frac{3a^2}{8}\int\limits_0^{\frac{\pi}{2}}(\sin^2 2t-\sin^2 2t\cdot\cos 2t)dt = \frac{3a^2}{8}\int\limits_0^{\frac{\pi}{2}}\left(\frac{1-\cos 4t}{2}-\sin^2 2t\cdot\cos 2t\right)dt =$$

$$= \frac{3a^2}{16}\int\limits_0^{\frac{\pi}{2}}(1-\cos 4t)dt - \frac{3a^2}{8}\int\limits_0^{\frac{\pi}{2}}\sin^2 2t\cdot\frac{1}{2}d(\sin 2t) = \frac{3a^2}{16}\left(t-\frac{1}{4}\sin 4t\right)\Big|_0^{\frac{\pi}{2}} - \frac{3a^2}{16}\cdot\frac{\sin^3 2t}{3}\Big|_0^{\frac{\pi}{2}} = \frac{3a^2\pi}{32}$$

ga teng ekanligi kelib chiqadi.

U holda $\quad x_c = y_c = \dfrac{M_y}{M} = \dfrac{8a^3\cdot 32}{105\cdot 3\pi a^2} = \dfrac{256a}{315\pi}.$

Demak, figuraning og'irlik markazi

$$C = \left(\frac{256a}{315\pi},\frac{256a}{315\pi}\right) \text{ ekan.}$$

6-misol. *ρ=a(1+cosθ)* kardioida bilan chearalangan figura og'irlik markazining dekart koordinatalarini toping (159-chizma).

Yechish. Figuraning *0x* o'qqa simmetrikligidan $y_c=0$ ekanligi kelib chiqadi. (45.12)formulalarning birinchisiga ko'ra:

$$x_c = \frac{2}{3}\cdot\frac{a^3\int\limits_0^{2\pi}(1+\cos\theta)^3\cos\theta d\theta}{a^2\int\limits_0^{2\pi}(1+\cos\theta)^2 d\theta}.$$

Suratdagi integralni hisoblaymiz:

$$\int_0^{2\pi}(1+\cos\theta)^3\cos\theta d\theta = \int_0^{2\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\cos\theta d\theta = \int_0^{2\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)d\theta =$$

$$= \int_0^{2\pi}\cos\theta d\theta + 3\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta + 3\int_0^{2\pi}\cos^2\theta\cos\theta d\theta + \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \sin\theta\Big|_0^{2\pi} + \frac{3}{2}(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{2\pi}+$$

$$+3\int_0^{2\pi}(1-\sin^2\theta)d\sin\theta + \frac{1}{4}\int_0^{2\pi}(1+2\cos 2\theta+\cos^2 2\theta)d\theta = 3\pi + 3(\sin\theta-\frac{\sin^3\theta}{3})\Big|_0^{\pi} + \frac{1}{4}(\theta+2\cdot\frac{1}{2}\sin 2\theta)\Big|_0^{2\pi}+$$

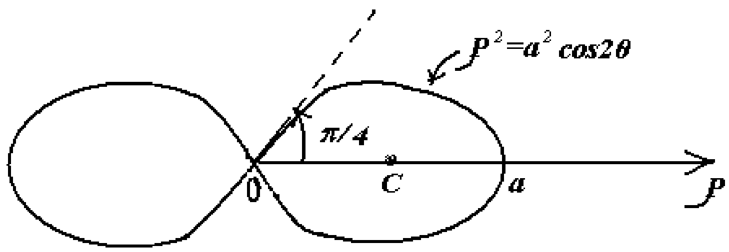
$$+\frac{1}{4}\int_0^{2\pi}\cos^2 2\theta d\theta = 3\pi+\frac{\pi}{2}+\frac{1}{8}\int_0^{2\pi}(1+\cos 4\theta)d\theta = \frac{7\pi}{2}+\frac{1}{8}(\theta+\frac{1}{4}\sin 4\theta)\Big|_0^{2\pi} = \frac{7\pi}{2}+\frac{\pi}{4}=\frac{15\pi}{4}.$$

Maxrajdagi integral kardioida bilan chegaralangan figuraning yuzini ikkilangani bo'lgani uchun u $3a^2\pi$ ga teng (42-ma'ruzadagi 8-misolga qarang). Demak,

$$x_c = \frac{2}{3}\cdot\frac{a^3\cdot\frac{15\pi}{4}}{3a^2\pi} = \frac{5a}{6}.$$

Shunday qilib berilgan figuraning og'irlik markazi $C(\frac{5a}{6},0)$ bo'lar ekan.

7-misol. $\rho^2=a^2(\cos 2\theta)$ Bernulli lemniskatasining o'ng sirtmogi bilan chegaralangan figura og'irlik markazining dekart koordinatalarini toping (184-chizma).



184-chizma.

Yechish. $\theta=0$ bo'lganda qutb radiusi ρ eng katta qiymati a ga teng bo'ladi. $\theta=\frac{\pi}{4}$ da $\rho=0.\theta$ qutb burchak $-\frac{\pi}{4}$ dan $\frac{\pi}{4}$ gacha o'zgarganda $M(\rho,\theta)$ nuqta lemniskataning o'ng sirtmogini chizadi. Qutb burchak θ $\frac{3\pi}{4}$ dan $\frac{5\pi}{4}$ gacha o'zgarganda $M(\rho,\theta)$ nuqta harakatlanib lemniskataning chap sirtmogini chizadi. Figura ox (ρ) o'qqa nisbatan simmetrik joylashganligi sababli $y_c=0$.

$\rho^3=a^3(\cos 2\theta)^{\frac{3}{2}}$ ga ega bo'lamiz. Integrallash chegaralari $\alpha=-\frac{\pi}{4}$ va $\beta=\frac{\pi}{4}$.

(45.12) formulaga binoan:

$$x_c=\frac{2}{3}\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^3(\cos 2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos 2\theta d\theta}=\frac{2}{3}\cdot\frac{a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta d\theta}=\frac{2}{3}a\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\frac{1}{2}\sin 2\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}=$$

$$=\frac{2}{3}a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}d(\sin\theta)\left|\begin{matrix}\sin\theta=\frac{1}{\sqrt{2}}\sin t,\cos\theta d\theta=\frac{1}{\sqrt{2}}\cos tdt\\ \theta=-\frac{\pi}{4}da\quad t=-\frac{\pi}{2},\theta=-\frac{\pi}{4}da\quad t=\frac{\pi}{2}\end{matrix}\right|=\frac{2}{3}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2 t)^{\frac{3}{2}}\cdot\frac{1}{\sqrt{2}}\cos tdt=$$

$$=\frac{2}{3\sqrt{2}}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3 t\cos tdt=\frac{2a}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 t)^2dt=\frac{2a}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dt=$$

$$=\frac{2a}{3\sqrt{2}}\cdot\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2t+\cos^2 2t)^2dt=\frac{a}{6\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2t+\frac{1}{2}(1+\cos 4t))dt=$$

$$=\frac{a}{6\sqrt{2}}(t+2\cdot\frac{1}{2}\sin 2t+\frac{1}{2}t+\frac{1}{2}\cdot\frac{1}{4}\sin 4t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{a}{6\sqrt{2}}(\pi+\frac{\pi}{2})=\frac{\pi a}{4\sqrt{2}}=\frac{\pi a\sqrt{2}}{8}.$$

Shunday qilib Bernulli lemniskatasining o'ng sirtmogi bilan chegaralangan figuraning og'irlik markazini dekart koordinatalari $x_c=\frac{\pi a\sqrt{2}}{8}$, $y_c=0$ ekan.

8-misol. Yarim o'qlari *a* va *b* bo'lgan ellipsning uning har ikkala o'qiga nisbatan inersiya momentini toping.

Yechish. Ellipsning $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamasidan $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$ ni topamiz. Ellips koordinata o'qlariga nisbatan simmetrik bo'lganligidan ellips bilan chegaralangan figura yuzining to'rtdan birining inersiya momentini hisoblab, natijani *4* ga ko'paytirish kifoya.

$y=\frac{b}{a}\sqrt{a^2-x^2}$, $x=0$, $y=0$ $(0\le x\le a)$ chiziqlar bilan chegaralangan figuraning $0x$ o'qqa nisbatan inersiya momentini topamiz. (45.10) formulaga binoan:

$$\frac{1}{4}\cdot I_x=\frac{1}{3}\int_a^b y^3dx=\frac{1}{3}\int_0^a(\frac{b}{a}\sqrt{a^2-x^2})^3dx=\frac{b^3}{3a^3}\int_0^a(\sqrt{a^2-x^2})^3dx\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=0\ \ da\ \ t=0,\\ x=a\ \ da\ \ t=\frac{\pi}{2}.\end{array}\right|=$$

$$=\frac{b^3}{3a^3}\int_0^{\frac{\pi}{2}}(\sqrt{a^2-a^2\sin^2 t})^3 a\cos tdt=\frac{b^3}{3a^3}\int_0^{\frac{\pi}{2}}(a\cos t)^3 a\cos tdt=\frac{b^3a}{3}\int_0^{\frac{\pi}{2}}\cos^4 tdt=\frac{b^3a}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dt=$$

$$=\frac{b^3a}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2t+\cos^2 2t\right)dt=\frac{b^3a}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt=\frac{b^3a}{12}(t+2\cdot\frac{1}{2}\sin 2t+\frac{1}{2}t+\frac{1}{2}\cdot\frac{1}{4}\sin 4t)\Big|_0^{\frac{\pi}{2}}=$$

$$=\frac{b^3a}{12}(\frac{\pi}{2}+\frac{\pi}{4})=\frac{b^3a}{12}\cdot\frac{3\pi}{4}=\frac{b^3a\pi}{16}.$$

Demak, $I_x=\frac{b^3a\pi}{16}\cdot 4=\frac{a\pi b^3}{4}.$

Xuddi Shunga o'xshash (45.10) formuladan foydalanib $I_y=\frac{\pi a^3 b}{4}$.ni ham topish mumkin.

Olingan natijalardan *a=b* bo'lganda *a* radiusli bir jinsli *(δ=1)* doiraning uning diametriga nisbatan inersiya momentini hosil qilamiz: $I=\frac{\pi a^4}{4}$.

9-misol. *R* radiusli bir jinsli (zichlik *δ=1*) doiraning qutb inersiya momentini, ya'ni doiraning uning markaziga nisbatan inersiya momentini toping.

Yechish. *m* massali *M* moddiy nuqtaning *P* nuqtaga nisbatan inersiya momenti deb $I_o=md^2$ songa aytiladi, bu yerdagi *d* *M* va *P* nuqtalar orasidagi masofa. Doira markazidan *ρ* masofada yotuvchi *dρ* qalinlikdagi doiraviy halqani olamiz, bunda *dρ* istalgancha kichik miqdor. Shartga ko'ra halqa bir jinsli bo'lganligi sababli uning massasi halqaning yuzi $\pi(\rho+d\rho)^2-\pi\rho^2\approx 2\pi\rho\, d\rho$ ga teng. Demak halqaning doira markaziga nisbatan inersiya momenti

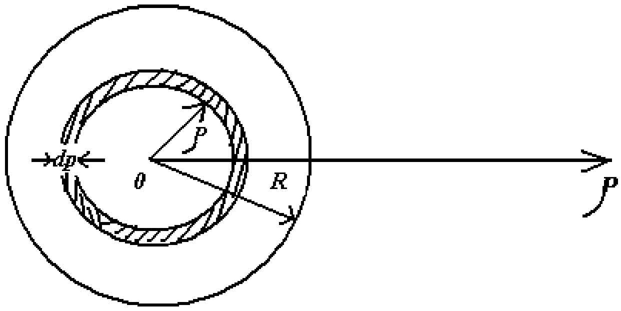
$$dI_o\approx \rho^2\cdot 2\pi\rho\, d\rho=2\pi\rho^3\, d\rho$$

va doiraning uning markaziga nisbatan inersiya momenti:

$$I_0=2\pi\int_o^R\rho^3d\rho=2\pi\cdot\frac{\rho^4}{4}\Big|_o^R=\frac{\pi R^4}{2}=\frac{MR^2}{2},$$

bu yerda $M=\pi R^2$ -doira massasi (yuzi).

Biz yuqorida keltirgan misollarimizda qaralayotgan geometrik figuralar bir jinsli, ya'ni chiziqli zichlik birga teng deb faraz qildik.



185-chizma.

O’z-o’zini tekshirish uchun savollar

1. O’qqa nisbatan statik moment nima?
2. Nuqtaga va o’qqa nisbatan inersiya momenti nima?
3. Moddiy nuqtalar sistemasining og’irlik markazi koordinatalarini topish formulalarini yozing.
4. Moddiy nuqtalar sistemasining koordinata o’qlariga nisbatan statik momentlarini topish formulalarini yozing.
5. Moddiy nuqtalar sistemasining koordinatalar boshiga nisbatan inersiya momentini topish formulasini yozing.
6. Moddiy nuqtalar sistemasining koordinata o’qlariga nisbatan inersiya momentalrini topish formulalarini yozing.
7. Tekislikda chiziq $y=f(x)$ tenglama bilan va parametrik tenglamalar hamda u qutb koordinatalar sistemasida berilaganda uning koordinata o’qlariga nisbatan statik va inersiya momentlari hamda og’irlik markazi qanday topiladi?
8. $y=f_1(x)$, $y=f_2(x)$, $x=a$, $x=b$ $(y=f_1(x)\leq y=f_2(x),\ a<b)$ chiziqlar bilan chigaralangan tekis figurav og’irlik markazi hamda uning o’qlarga nisbatan statik va inersiya momentlari qanday topiladi?
9. $\theta=\alpha$, $\theta=\beta$ $(\alpha<\beta)$ nurlar va qutb koordinatalar sistemasida $\rho=\rho(\theta)$ tenglamaga ega egri chiziq bilan chegaralangan egri chiziqli sektorning og’irlik markazini abssissasi va ordinatasini hamda koordinata o’qlari $0x$, $0y$ larga nisbatan statik momentini topish formulalarini yozing.
10. $y=f(x)$, $y=0$, $x=a$, $x=b$ chiziqlar bilan chegaralangan bir jinsli figura og’irlik markazining koordinatalari hamda uning koordinata o’qlariga nisbatan statik va inersiya momentlari qanday topiladi?

Mustaqil yechish uchun mashqlar

1. $y=2\sqrt{x}$ egri chiziq yoyining $x=3$ to’g’ri chiziq kesgan qismining $0x$ o’qqa nisbatan statik momentini toping. Javob: $M_x=\frac{28}{3}$.

2. $x=a\,(t-\sin t)$, $y=a(1-\cos t)$ sikloida birinchi arkasining og’irlik markazi koordinatalarini toping. Javob: $C(\pi a;\frac{4a}{3})$

3. $\rho=4\cos\theta$ aylananing qutbdan qutb o’qiga perpendikulyar bo’lib o’tgan to’g’ri chiziqqa nisbatan statik momentini toping. Javob: 8π.

4. $x^2+4y-16=0$ parobola va $0x$ o'q bilan chegaralangan bir jinsli figuraning og'irlik markazi topilsin. Javob: $C(0; 8/5)$.

5. $\rho=2(1-\cos\theta)$ kardioida bilan chegaralangan figura og'irlik markazining dekart koordinatalarini toping. Javob: $C(\frac{5}{3};0)$

6. $y=4\sqrt{x}$ parabola yoyini $x=4$ to'g'ri chiziq kesgan qismining abssissalar o'qiga nisbatan inersiya momentini toping. Javob: $32(6\sqrt{2}-\ell n(3+2\sqrt{2}))$.

7. $y=2-x^2$ va $y=x^2$ chiziqlar bilan chegaralangan figuraning koordinata o'qlariga nisbatan inersiya momentini toping. Javob: $I_y=\frac{8}{15}$, $I_x=\frac{356}{105}$.

46-ma'ruza. Mavzu: Aniq integralning fizika masalalarini yechishga tatbiqlari

Reja:

1. O'zgaruvchan tezlkka ega nuqtaning bosib o'tgan yo'li.
2. O'zgaruvchan kuchning bajargan ishi.
3. Suyuqlikning bosim kuchini hisoblash.
4. Kinetik energiya.

Adabiyotlar: 1,2,3,4,5,6,7,9,12.

Tayanch iboralar: tezlik, yo'l, tezlanish, ish, bosim, kinetik energiya.

46.1. O'zgaruvchan tezlikka ega nuqtaning bosib o'tgan yo'li

Aniq integralning geometriya va mexaniqaga tatbiqlarida ko'rdikki u yoki bu masalalarni yechish uchun berilgan geometrik figura n ta ixtiyoriy qismlarga ajratilib qo'yilgan masala avval figuraning bitta qismi (elementar bulagi) uchun hal etiladi. Keyin olingan natijani jamlab integral yig'indi tuziladi. Integral yig'indida limitga o'tilsa qo'yilgan masalani yechish uchun aniq formula chiqarildi.

Bundan buyon fizika masalalarini yechishda ham shu usuldan foydalanamiz.

Faraz qilaylik nuqta o'zgaruvchan v tezlik bilan to'g'ri chiziqli harakat qilayotgan bo'lsin. v tezlik t vaqtning ma'lum funksiyasi, ya'ni $v=f(t)$ bo'lsin. Nuqtani vaqtning t_0 momentidan T mamentigacha bosib o'tgan yo'lini aniqlash talab etilsin. $[t_0, T]$ oraliqni n ta ixtiyoriy $[t_0, t_1],[t_1, t_2],\dots, [t_{i-1},t_i],\dots, [t_{n-1}, t_n]$ $(t_n=T)$ qismlarga ajratamiz. Har bir $[t_{i-1},t_i]$ $(i=\overline{1,n})$ bo'lakda ixtiyoriy z_i nuqta olib bu oraliqda nuqtaning tezligi o'zgarmas va u $f(z_i)$ ga teng deb faraz qilamiz. U holda nuqtaning $\Delta t_i= t_i- t_{i-1}$ vaqt oralig'ida bosib o'tgan yo'li ΔS_i taqriban $f_i(z_i)\,\Delta t_i$ ga teng bo'lishi ayon. Nuqtaning butun $[t_0,T]$ oraliqda o'tgan yo'li

$$S \approx \sum_{i=1}^{n} f(z_i)\Delta t_i$$

bo'ladi. Bu yig'indi $[t_0,T]$ kesmada $f(t)$ funksiya uchun integral yig'indi ekanligini hisobga olib oxirgi taqribiy tenglikda $\max \Delta t_i \to 0$ da limitga o'tsak

$$S=\int_{t_0}^{T} v(t)dt = \int_{t_0}^{T} f(t)dt$$

kelib chiqadi. Bu $[t_0,T]$ vaqt oralig'ida nuqtaning bosib o'tgan yo'lini topish formulasidir.

Eslatma. $[t_0,T]$ oraliqda o'zgarmas v tezlik bilan to'g'ri chiziqli harakat qilayotgan nuqtaning shu vaqt oralig'ida bosib o'tgan yo'li.

$$S=v(T-t_0)$$

kabi aniqlanishi maktab kursidan ma‘lum.

1-misol. Nuqtaning tezligi $v=(3t^2+2t+1)$ *m/s* ga teng. Harakat boshlangandan so'ng o'tgan $t=10$ *s* ichida nuqta bosib o'tgan S yo'lini toping.

Yechish. Shartga ko'ra $f(t)= 3t^2+2t+1,\ t_0=0,\ T=10.$ (46.1)formulaga binoan

$$S=\int_0^{10}(3t^2+2t+1)dt=(t^3+t^2+t)\Big|_0^{10}=1110(m).$$

2-misol. Nuqtaning tezligi $v=(9t^2-8t)$ *m/s* ga teng. Nuqtaning *4*-sekundda bosib o'tgan S yo'lini toping.

Yechish. Shartga ko'ra $f(t)= 9t^2-8t,\ t_0=3,\ T=4.$

Demak,

$$S=\int_3^4(9t^2-8t)dt=\left(9\cdot\frac{t^3}{3}-8\cdot\frac{t^2}{2}\right)\Big|_3^4=(3t^3-4t^2)\Big|_3^4=83(m)\,.$$

3-misol. Nuqtaning tezligi $v=(12t-3t^2)$ *m/s* ga teng. Nuqtaning harakat boshlanganidan uning to'xtagunicha o'tgan vaqt oralig'ida bosib o'tgan S yo'li topilsin.

Yechish. Nuqtaning tezligi harakat boshlangunicha va nuqta to'xtaganida nolga tengligini hisobga olib $12t-3t^2=0$ yoki $t(4-t)=0$ tenglamani yechib $t_0=0,t=4$ ekanini topamiz.

Demak, $S=\int_0^4(12t-3t^2)dt=(6t^2-t^3)\Big|_0^4=32(m)$

4-misol. Jism yer sathidan yuqoriga vertikal yo'nalishda $v=(39{,}2-9{,}8t)$ *m/s* tezlik bilan otildi. Shu jism yerdan qancha balandlikka ko'tarilishini toping.

Yechish. Jism yerdan eng yuqoriga ko'tarilganda uning tezligi $v=o$ ya'ni $39{,}2-9{,}8t=0,$ bundan $t=4s$ kelib chiqadi. Shuning uchun *(46,1)* formulaga binoan

$$S=\int_0^4 (39{,}2-9{,}8t)dt=(39{,}2t-4{,}9t^2)\Big|_0^4=78{,}4(m)$$

kelib chiqadi.

5-misol. *48km/soat* tezlik bilan harakatlanayotgan avtomobil tormoz berib tezligini kamaytira boshladi va *3* sek. dan keyin to'xtadi. Avtomobil butunlay to'xtaguncha qancha masofani bosib o'tishini toping (ishqalanishni va havoning qarshiligini hisobga olmang).

Yechish. Tekis sekinlanuvchan harakatning tezligi

$$v=v_0-at$$

formula orqali topiladi, bu yerda v_0-boshlangich tezlik, a-tezlanish.

Masalaning shartiga ko'ra $v_0=48\ km/soat=48\cdot\frac{1000m}{3600s}=\frac{40}{3}\cdot\frac{m}{s}$.

Tezlanish a ni avtomobil *3 sek.* dan keyin to'xtash shartidan, ya'ni $t=3$ sekunda $v=0$ shartdan topamiz: $0=\frac{40}{3}-a\cdot 3,\quad a=\frac{40}{9}\frac{m}{sek^2}$.

v_0 va a ning qiymatini tezlikning formulasiga qo'yib, topamiz: $v=\frac{40}{3}-\frac{40}{9}t.$

Demak, $S=\int_0^3\left(\frac{40}{3}-\frac{40}{9}t\right)dt=\left(\frac{40}{3}t-\frac{40}{9}\cdot\frac{t^2}{2}\right)\Big|_0^3=40-20=20\ (m)$.

6-misol. Reaktiv samolyot *20* sekund ichida o'z tezligini *360 km/soat* dan *720 km/soat* ga oshirdi. Samolyotning tezligini tekis tezlanuvchan deb hisoblab, u qanday tezlanish bilan uchganini va shu vaqt oralig'ida qancha masofani bosib o'tganini toping.

Yechish. Tekis tezlanuvchan harakat tezligi $v=v_0+at$ formula orqali ifodalanadi. Masalaning shartiga ko'ra

$t=0$ da $v_0=360\ km/soat=360\cdot\frac{1000\ m}{3600\ s}=100\frac{m}{s}$,

$t=20\ c$ da $v=720\ km/soat=720\cdot\frac{1000\ m}{3600\ s}=200\frac{m}{s}$.

v_0 va t ning qiymatlarini tezlikning formulasiga qo'yib, a tezlanishni topamiz:

$$200=100+a\cdot 20,\quad a=5m/sek^2.$$

Demak, samolyotning tezligi $v=(100+5t)m/s$ bo'ladi. (46.1) formulaga ko'ra

$$S=\int_0^{20}(100-5t)dt=(100t-5\cdot\frac{t^2}{2})\Big|_0^{20}=3000(m)=3\ km.$$

7-masala. *294m* balandlikdan pastga vertikal yo'nalishda *19,6 m/s* boshlanshgich tezlik bilan jism tushadi. Necha sekunddan keyin jism yerga kelib tushadi? (Og'irlik kuchi tezlanishi $g=9{,}8\ m/sek^2$).

Yechish. Erkin tushayotgan jism tezligi (havoning qarshiligini hisobga olmaganda) $v=v_0+gt$ formula orqali ifodalanadi. $v=19{,}6+9{,}8t$ ga egamiz.

Jismning tushish vaqti x ni $h=\int_0^x(19{,}6+9{,}8t)dt=(19{,}6t+9{,}8\cdot\frac{t^2}{2})\Big|_0^x=19{,}6x+4{,}9x^2$

tenglamadan topamiz, bu yerda $h=294\ m$. $4{,}9x^2+19{,}6x-294=0$ tenglamani yechamiz.

Uni *4,9* ga qisqartirsak $x^2+4x-60=0$ tenglama hosil bo'ladi. Bu tenglama

$x_{1,2}=-2\sqrt{4+60}=-2\pm 8$ ya'ni $x_1=-10$, $x_2=6$ ildizlarga ega. Shartga ko'ra $t>0$ ekanini hisobga olsak faqat $x=6$ masalaning yechimi bo'lishini ko'ramiz. Shunday qilib jism *t=6 sek* dan keyin yerga kelib tushar ekan.

46.2 O'zgaruvchan kuchning bajargan ishi.

M moddiy nuqta *F* kuch ta'siri ostida *0x* to'g'ri chiziq bo'ylab harakatlanayotgan bo'lsin va bunda kuchning yo'nalishi harakat yo'nalishi bilan bir xil bo'lsin (*F* kuch *0x* o'qqa parallel va ular bir xil yo'nalgan).

Shu *F* kuchning *M* moddiy nuqtani $x=a$ vaziyatdan $x=в$ vaziyatga ko'chirishda bajarogan ishi *A* ni topish talab etilsin.

Bunda ikki holatni kuzatish mumkin.

1. *F* kuch o'zgarmas bo'lsin. U holda nuqtani $x=a$ vaziyatdan $x=в$ vaziyatga ko'chirishda *F* kuchning bajargan ishi

$$A=F(b-a) \qquad (46.2)$$

formula yordamida topilishi ma'lum.

2. *F* kuch *M* nuqtaning vaziyatiga bog'liq ravishda o'zgarsin, ya'ni $[a,b]$ kesmada $F(x)$ uzluksiz funksiya bo'lsin. U holda *F* kuch bajargan *A* ishni quyidagicha topiladi (186- chizma).

$[a,b]$ kesmani $a=x_o< x_1< x_2<\ldots x_{i-1}< x_i<\ldots< x_n=b$ nuqtalar yordamida *n* ta ixtiyoriy $[x_{i-1}, x_i]$ $(i=\overline{1,n})$ mayda qismlarga ajratib har bir $[x_{i-1}, x_i]$ bo'lakda bittadan ixtiyoriy z_i nuqta olamiz. Uzunligi $\Delta x_i= x_i- x_{i-1}$ bo'lgan $[x_{i-1}, x_i]$ mayda bo'lakda *F* kuch o'zgarmas va u $F(z_i)$ga teng deb faraz qilamiz. U holda (46.2) formulaga ko'ra *F* kuchning $[x_{i-1}, x_i]$ oraliqda bajargan ishi

$$A\approx F(z_i)\,\Delta x_i$$

bo'ladi. Shunga o'xshash mulohazalarini har bir kesma uchun o'tkazib *F* kuchning $[a,b]$ kesmada bujargan ishi *A* ning taqribiy qiymati

$$A\approx\sum_{i=i}^{n}F(z_i)\Delta x_i$$

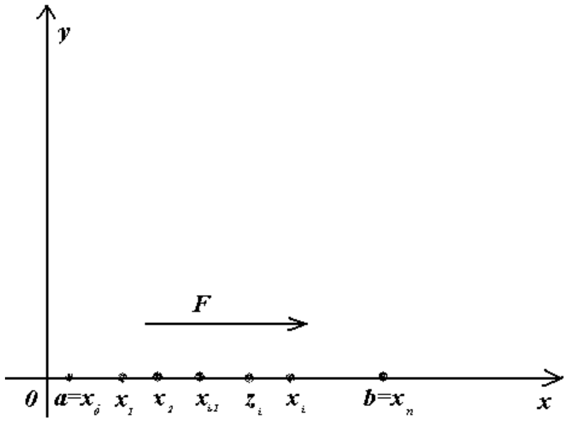
ni hosil qilamiz. Bu tenglikning o'ng tomonidagi yig'indi $[a,b]$ kesmada uzluksiz $F(x)$ funksiya uchun integral yig'indi bo'ladi. Shuning uchun u

$\lambda=max\Delta x_i\to 0$ da aniq limitga ega va $F(x)$ funksiyadan $[a,b]$ oraliq bo'yicha olingan aniq integralga teng, ya'ni

$$A=\int_a^b F(x)dx\,.$$

8-misol. Yer sathidan vertikal yo'nalishda *m* massali jismni *h* balandlikka chiqarish uchun zarur bo'lgan kuchning bajargan A ishi topilsin (187-chizma).

Yechish. Yerning tortish kuchini F_i massasini m_y, jismdan yerning markazigacha masofani x desak Nyuton qonuniga ko'ra $F = G\frac{m \cdot m_y}{x^2}$ bo'ladi. Agar $Gm \cdot m_y = K$ belgilashni kiritsak $F(x) = \frac{K}{x^2}$ ga ega bo'lamiz, bunda $R \leq x \leq h + R$ R-yerning radiusi. $x = R$ da $F(R)$kuch jismning og'irlik kuchi $P = m \cdot g$ ga teng va $\frac{K}{x^2} = P$, $K = PR^2$, $F(x) = \frac{PR^2}{x^2}$.



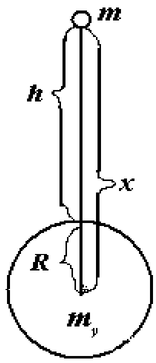
186-chizma.

Buni (46.3) formulaga qo'yib quyidagini hosil qilamiz.

$$A = \int_R^{R+h} F(x)dx = PR^2 \int_R^{R+h} \frac{dx}{x^2} = -PR^2 \frac{1}{x}\Big|_R^{R+h} = -\frac{PR^2}{R+h} + \frac{PR^2}{R} = \frac{PRh}{R+h}.$$

9-misol. Ikkinchi kosmik tezlik topilsin.

Yechish. Jismning ikkinchi kosmik tezligini ya'ni (187-chizma) jism yerning tortish maydonidan planetalararo fazoga chiqishi uchun u qanday boshlangich tezlikka ega bo'lishi kerak degan savolga javob izlaymiz. 8-misolning natijasidan hamda undagi belgilashlardan foydalanamiz.



187-chizma.

Jismning planetalararo fazoga chiqishi uni cheksiz balandlikka *(h=∞)* chiqishni anglatadi. Shuning uchun

$$A = \frac{PRh}{R+h}$$

tenglikda da limitga o'tib quyidagini hosil qilamiz.

$$\lim_{h\to\infty} A = \lim_{h\to\infty} \frac{PR}{\frac{R}{h}+1} = PR = mgR,$$

bunda g-jismning yer sathiga erkin tushish tezlanishi. Bu ish jismning kinetik energiyasining o'zgarishi hisobiga amalga oshiriladi. Shuning uchun boshlangich momentda jismning kinetik energiyasi shu ishdan kichik bo'lmasligi, ya'ni jismning boshlangich tezligi v shunday bo'lishi kerakki $\frac{mv^2}{2} \geq mgR$ bo'lsin.

Bundan $v \geq 2gR$, $v \geq \sqrt{2gR} = \sqrt{2 \cdot 10 \cdot 6400000} m/s = 1{,}4 \cdot 8000 m/s = 11{,}2\ km/s.$

Agar jismning boshlangich tezligi *11,2 km/s* ga teng bo’lsa jism parabola bo’ylab harakatlanadi. Jismning boshlangich tezligi *11,2 km/*s dan katta bo’lganda jism giperbola bo’ylab harakat qiladi. Jismning boshlangich tezligi *11,2 km/s* dan kichik bo’lganda u ellips bo’ylab harakat qiladi. Shuning uchun bu holda jism yo yerga qulab tushadi yoki yerning sun‘iy yo’ldoshiga aylanadi.

10-misol. Agar prujinani *1 sm* ga qisish uchun *10 n* kuch kerak bo’lsa, uni *4 sm* ga qisish ga sarf bo’ladigan *F* kuch bajaradigan ishni toping.

Yechish. Guk qonuniga muvofiq *F* kuch va *x* siljish o’zaro *F=kx* munosabat bilan bog’langan (*k*-proporsionallik koeffitsienti). *k* ni masala shartidan topamiz: *x=1 sm=0,01 m* da kuch *F=10 n*, ya‘ni *10=κ·0,01* bundan *κ=1000 n/m*. Demak(46.3) ko’ra

$$A=\int_0^{0,04}1000x\,dx=500x^2\Big|_0^{0,04}=500n/m\cdot 0,0016m^2=0,8(j).$$

11-misol. Uzunligi *1m*, kesimining radiusi *2mm* bo’lgan mis simni *1mm* ga cho’zishda bajarilgan ishni hisoblang.

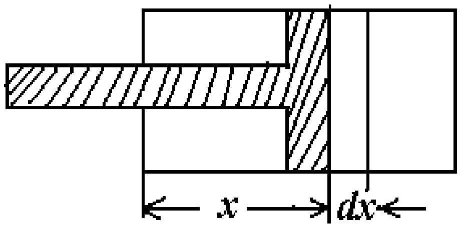
Yechish. Uzunligi *lm* va kesimining yuzi *Smm*^{2} bo’lgan simni *x m* ga cho’zish uchun zarur *F* kuch $F=E\frac{Sx}{l}$ formula orqali ifodalanadi, bu yerda *E*-elastiklik moduli. Mis uchun *E* ni *E=120000 n/mm*^{2} deb olamiz. U holda

$$A=\int_0^{0,001}E\cdot\frac{sx}{\ell}dx=\frac{ES}{\ell}\int_0^{0,001}x\,dx=\frac{E\cdot S}{\ell}\cdot\frac{x^2}{2}\Big|_0^{0,001}=\frac{ES}{2\ell}\cdot(0,001)^2.$$

Bunga *E=120000 n/mm*^{2}, *S=2πR=2π·2=4 πmm*^{2}, *ℓ=1m* qiymatlarni qo’yib bajarilgan ishni topamiz:

$$A=120000\cdot 4\pi\cdot\frac{0,000001}{2\cdot 1}=0,24\pi(j).$$

12-misol Ko’ndalang kesimining yuzi *S kv.* birlik bo’lgan harakatlanuvchi porshenga ega silindr gaz bilan to’ldirilgan. Gazning hajmi oshganda silindrda Boyl-Mariotta qonuni *pv=k=const* saqlanadi deb hisoblab gazning bosim kuchi ta‘sirida uning hajmi v_0 dan v_1 gacha o’zgarganda shu kuchning bajargan ishi *A* topilsin (gazning harorati o’zgarmaydi).



188-chizma.

Yechish. *x(m)*-porshenning o’tgan masofasi bo’lsin (188-chizma).

x juda kichik *dx* ga o’zgarganda gazning bosimi o’zgarmaydi hajmi *v* esa *Δv* ga o’zgaradi deb faraz qilamiz. U holda bosim kuchining *dx* kesmada bajargan ishi *ΔA* quyidagi taqribiy qiymat yordamida ifodalanadi, ya‘ni

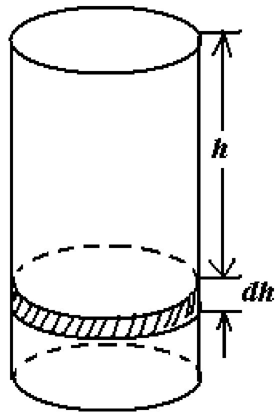
$\Delta A \approx P \cdot sdx.\ \ P=\frac{k}{v}$ va *sdx=Δv* (silindrning hajmi asosining yuzi *s* bilan balandligi *dx* ning ko'paytmasiga teng) ekanini hisobga olsak $\Delta A \approx \frac{k}{v}\Delta v = k\frac{\Delta v}{v}$ hosil bo'ladi. Bu yerdagi *ΔA, Δv* orttirmalarni *dA, dv* differensiallarga almashtirib

$$dA = k\frac{dv}{v}$$

tenglikka ega bo'lamiz. Buni v_0 dan v_1 gacha integrallab $A = k\ell n\frac{v_1}{v_0}$ ni hosil qilamiz.

13-misol. Asosining radiusi *R=3m,* balandligi *P=5* m bo'lgan silindrik idishdagi suvni tortib chiqarish uchun kerak bo'ladigan ishni hisoblang (189-chizma).

Yechish. Birorta jismni ko'tarishga sarflanadigan kuchning bajargan *A* ishning kattaligi jismni ko'tarish balandligi *h* ga bog'liq bo'ladi, ya'ni *A=Ph* bu yerda
P- jismning og'irligi.
Ma'lumki, idishdan *h* balandlikdagi suv qatlamini tortib chiqarish uchun sarf bo'ladigan kuchning bajargan ishi *h* ning funksiyasi, ya'ni *A(h)* bo'ladi. *h* miqdor *dh* kattalikka ortganda suv hajmi $\Delta v=\pi R^2 dh$ ga (silindrning hajmi asosining yuzi bilan balandligining ko'paytmasiga teng), uning og'irligi *P,* $\Delta P=\pi\delta R^2 dh$ $(\Delta P=\delta\cdot\Delta v)$ kattalikka (bu yerda *δ-1*-suvning solishtirma og'irligi), ish esa $\Delta A=\pi\delta r^2 hdh$ kattalik ortadi. *ΔA* orttirmani *dA* differensialga almashtirib



189-chizma.

$$dA=\pi\delta R^2 hdh$$

tenglikka ega bo'lamiz. Bu tenglikni *h=0* dan *h=H* gacha integrallab butun *A* ishni topamiz: $A=\int_0^H \pi\delta R^2 hdh = \frac{\pi\delta R^2}{2}\cdot h^2\Big|_0^H = \frac{\pi\delta R^2 H^2}{2}$, bunga $\delta=1 tonna/m^3=10000$ n/m^3 *R=3m, H=5 m* qiymatlarni qo'ysak $A=\frac{1}{2}\pi\cdot 10000\cdot 9\cdot 25 = 1125000\,(j)$.

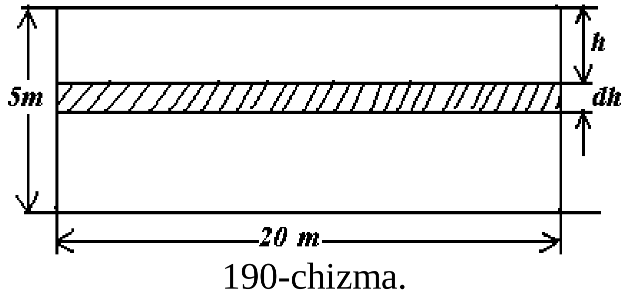
46.3. Suyuqlikning bosim kuchini hisoblash

Suyuqlikning bosim kuchini hisoblash uchun Paskal qonunidan foydalaniladi, unga ko'ra cho'kish (botish) chuqurligi *h* bo'lgan *s* yuzga suyuqlikning bosim kuchi

$$P=\gamma hs$$

ga teng, bu yerda *γ*-suyuqlikning solishtirma og'irligi.

14-misol. Vertikal to'g'on asosi *20m* va balandligi *5m* bo'lgan to'g'ri to'rtburchak shaklida (suvning sathi to'g'onning yuqori asisi bilan barobar), suvning butun to'g'onga bosim kuchini toping (190-chizma).



190-chizma.

Yechish. Paskal qonuniga muvofiq:

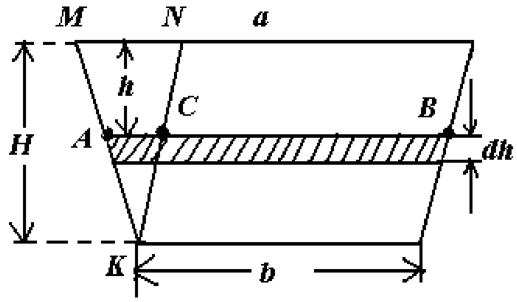
P=γhs=9,807 hs(n) (suv uchun *γ=1000kg/m³=1000·9,807kg/m³*) ya‘ni bosim kuchi *h* chuqurlikning birorta *p(h)* funksiyasidan iborat. Eni juda kichik *dh* ga teng shtrixlanagan to’g’ri to’rtburchakni olib uni *h* chuqurlikda gorizontal joylashgan deb faraz qilamiz. U holda bu bo’lakchaga bo’lgan bosim

$$dh=9807h\cdot 20dh=9807\cdot 20hdh$$

bo’ladi. Buni 0 dan 5 gacha integrallab suvning butun to’g’onga bosim kuchini topamiz:

$$P = 9807\cdot 20\int_0^5 hdh = 9807\cdot 10\cdot h^2\Big|_0^5 = 9807\cdot 250(n) = 2451750(n) = 2,45(mn).$$

15-misol. Vertikal to’g’on teng yonli trapetsiya Shaklida bo’lib, yuqori asosi *a=6,4 m,* pastki asosi *b=4,8 m,* balanligi esa *H=3 m.* Suvning butun to’g’onga bosim kuchini toping (191-chizma).



191-chizma.

Yechish. Trapetsiyaning shtrixlangan bo’lakchasi *h* chuqurlikda gorizontal joylashgan va u tomonlari *AB* va *dh* bo’lgan to’g’ri to’rtburchakdan iborat deb faraz qilamiz. U holda bu bo’lakka bo’lgan suvning bosimi.

$$dp=9807hABdh=9807h(AC+CB)dh=9807h(AC+b)dh \quad (n)$$

bo’ladi. *AC* ni *KAC* va *KMN* uchburchaklarning o’xshashligidan topamiz:

$$\frac{AC}{MN} = \frac{H-h}{H}, \quad \frac{AC}{a-b} = \frac{H-h}{H}, \qquad AC = \frac{a-b}{H}(H-h).$$

Bu ifodani *h* bo’yicha *0* dan *H* gacha integrallab, butun to’g’onga ta‘sir etayotgan bosim kuchini topamiz:

$$P = 9807\cdot\frac{1}{H}\int_0^H h(aH - h(a-b)dh) = 9807\cdot\frac{1}{H}\int_0^H \left[aHh - (a-b)h\right]dh = 9807\cdot\frac{1}{H}\left(\frac{aHh^2}{2} - \frac{(a-b)h^3}{3}\right)\Big|_0^H =$$

$$= 9807\cdot\frac{1}{H}\left(\frac{aH^3}{2} - \frac{(a-b)H^3}{3}\right) = 9807\cdot\frac{H^3}{H}\left(\frac{3a-2a+2b}{6}\right) = 9807\cdot\frac{H^2(a+2b)}{6}.$$

bunga *H=3 m, a=6,4 m, b=4,8 m* qiymatlarni qo’yib, topamiz:

$$P = 9807\cdot\frac{9(6,4+4,8\cdot 2)}{6} = 9807\cdot 24 = 235368 \quad (n).$$

46.6 Kinetik energiya

Massasi *m* ga, tezligi *v* ga teng bo’lgan moddiy nuqtaning kinetik energiyasi deb

$$k = \frac{mv^2}{2}$$

kattalikka aytiladi.

Massalari $m_1, m_2, \ldots, m_n$, tezliklari mos ravishda $v_1, v_2, \ldots, v_n$, larga teng bo’lgan *n* ta moddiy sistemasining kinetik energiyasi

$$K = \sum_{i=1}^{n} \frac{m_i v_i^2}{2}$$

ga tengdir.

Moddiy jism (figura)ning kinetik energiysini ham yuqorida qaralgan masalalarni yechishda foydalanilgan usuldan foydalanib topamiz, ya'ni berilgan jismni n ta kichik (elementar) qismlarga ajratib ularni moddiy nuqtalar sistemasi deb qaraymiz va ularni kinetik energiyalarini jamlab qandaydir funksiyaning integral yig'indisiga ega bo'lamiz. Unda limitga o'tib, qiymati jismning izlanayotgan kinetik energiyasiga teng bo'lgan aniq integralni hosil qilamiz.

16-misol. Massasi M va radiusi R bo'lgan disk uning markazidan disk tekisligiga perpendikulyar bo'lib o'tgan o'q atrofida ω burchak tezlik bilan aylanyapti. Uning kinetik energiyasini hisoblang.

Yechish. Diskning radiuslari $0<r_1<r_2<r_3<...<r_{i-1}<r_i<<r_n=R$ bo'lgan aylanalar yordamida n ta ixtiyoriy halqalarga ajratamiz. Qalinligi $\Delta r_i = r_i - r_{i-1}$ $(i=\overline{1,n})$ bo'lgan halqani qaraymiz. Bu halqaning massasi

$$\Delta m_i = \rho\Delta s_i = \rho\pi(r_i^2 - r_{i-1}^2) = \rho\pi(r_i + r_{i-1})(r_i - r_{i-1}) = 2\pi\rho\frac{r_i + r_{i-1}}{2}\Delta r_i = 2\pi\rho\overline{r}_i\Delta r_i$$

bunda $\rho = \frac{M}{\pi R^2}$ -diskning zichligi, $\overline{r}$ $[r_i, r_{i-1}]$ kesmaning o'rtasi. U holda

$$\Delta m_i = 2\pi\overline{r}_i \cdot \frac{M}{\pi R^2}\Delta r_i = \frac{2\overline{r}_i M}{R^2}\Delta r_i.$$

Δm_i massaning chiziqli tezligi $v_i = \overline{r}_i\omega$ ga teng. Demak elementar kinetik energiya quyidagiga teng bo'ladi:

$$\Delta K_i = \frac{v_i^2\Delta m_i}{2} = \frac{(\overline{r}_i\omega)^2}{2}\cdot\frac{2\overline{r}_i M}{R^2}\Delta r_i = \frac{\omega^2 M}{R^2}\overline{r}_i^{\,3}\Delta r_i.$$

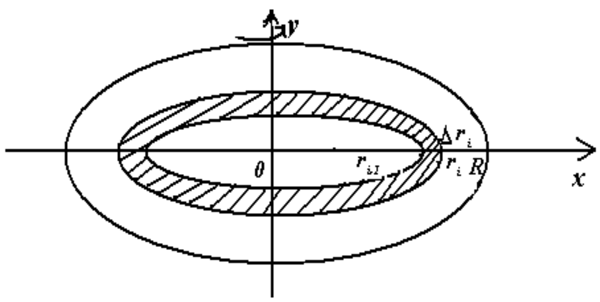
Barcha elementar kinetik energiyalarni jamlab

$$K \approx \sum_{i=1}^{n}\frac{\omega^2 M}{R^2}\overline{r}_i^{\,3}\Delta r = \frac{\omega^2 M}{R^2}\sum_{i=1}^{n}\overline{r}_i^{\,3}\Delta r$$

ga ega bo'lamiz. Bunda $\max\Delta r_i \to 0$ da limitga o'tsak

$$K = \frac{\omega^2 M}{R^2}\int_0^R r^3 dr = \frac{\omega^2 M}{R^2}\cdot\frac{r^4}{4}\Big|_0^R = \frac{\omega^2 MR^2}{4}$$

hosil bo'ladi.



192-chizma.

O’z-o’zini tekshirish uchun savollar

1.O’zgarmas tezlikda yo’lni topish formulasini yozing.
2. Aniq integral yordamida yo’lni topish formulasini yozing.
3. Tekis sekinlanuvchan harakatning tezligi qanday topiladi?
4. Tekis tezlanuvchan harakatnig tezligi qanday topiladi?
5. Nyutonning butun olam tortishish qonuni ayting?
6. O’zgaruvchan kuchning bajargan ishini topish formulasini yozing.
7. Guk qonunini ayting.
8. Paskal qonuni nimadan iborat?
9. Moddiy nuqtaning kinetik energiyasi nima va u qanday topiladi?
10. Moddiy jismning kinetik energiyasi qanday topiladi?

Mustaqil yechish uchun mashqlar

1. Nuqtaning tezligi $v=(100+8t)$ *m/sek.* Bu nuqta *[0;10]*vaqt oralig’ida qanday masofani bosadi. Javob: *1400 m*

2. Nuqtaning harakat tezligi $v=t\cdot e^{-0,01t}$ *m/sek* ga teng. Harakat boshlangan to’la to’xtagunga qadar nuqta bosib o’tgan yo’lni toping. Javob*: 10 km.*

3. Agar prujinani *1 sm* ga qisish uchun *10 n.* kuch kerak bo’lsa, prujinani *8 sm* ga qisish uchun sarf bo’ladigan *F* kuchning bajaradigan ishini toping. Javob: *3,2 (j)*

4. Prujinani *0,05 m* ga qisish uchun *25 j* ish sarflanadi. Prujinani *0,1 m* ga qisish uchun qancha sarflanadi? Javob: *100 j.*

5. Asosi *0,2 m* va balandligi *0,4 m* bo’lgan uchburchakli plastinka suvga shunday tik botirilganki uning uchi suvning sathida yotib asosi unga parallel. Suvning plastinkaga bosim kuchini toping. Javob: *104,6 (n).*

6. Markazi suyuqlikka (suyuqlikning solishtirma og’irligi γ ga teng) *h* chuqurlikda botirilgan *2a* va *2в* o’qli vertikal ellipsga (ellipsning katta o’qi *2a* suyuqlik
sathiga parallel, $h\geq b$) suyuqlikning bosim kuchini toping.
Javob: $p=ab\pi h$.

7. Diametr atrofida minutga *n* marta aylanayotgan *M* massali va *R* radiusli yarim aylananing kinetik energiyasini hisoblang. Javob: $\frac{\pi^2 n^2 R^2 M}{3600}$.